

State Estimation: Factor Graph for Integrated Navigation I AAE4203 – Guidance and Navigation

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GNSS Benefits

> GNSS provides a high long-term position accuracy with errors limited to a few meters (stand-alone), while user equipment is available for less than \$100 (€80).

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GPS Drawbacks

> One primary concern with using GPS as a standalone source for navigation is **signal interruption**.

If the number of usable satellites is less than three, some receivers have the option of not producing a solution or extrapolating the last position and velocity solution forward in what is called *deadreckoning* (DR) navigation. Inertial navigation systems (INSs) can be used as a flywheel to provide navigation during shading outages.

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GPS Drawbacks

> The **low update rate** of the GPS observations in some equipment is also of concern in real-time applications, especially those related to vehicle control.

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Framework of Sensors to Navigation

Sensor Integration Based on Kalman Filter

SANTA

- How the Sensor Fusion Problem Looks like…
- >GPS provide the position in x, y, z
- >IMU provide the linear angular velocity in x, y, z
- >GPS Doppler provide velocity in x, y, z

>Visual positioning provide relative motion Δx , Δy , Δz

How to achieve this sensor fusion by combining the positioning from different sources?

State Estimation Methods

STATE ESTIMATION FOR ROBOTICS

Basics for Probabilistic

Event A and B. $P(A)$ denotes the probability that the event A happens.

$$
P(B|A) = \frac{P(AB)}{P(A)}
$$
 $P(AB) = P(B|A)P(A)$ $P(B|A) = \frac{P(B|A)P(A)}{P(A)}$

Event z denotes the measurements. $P(z)$ denotes the probability that the event A happens.

$$
P(\mathbf{x}_k|\mathbf{z}_1, \dots, \mathbf{z}_k) = \frac{P(\mathbf{z}_0, \dots, \mathbf{z}_k|\mathbf{x}_k)P(\mathbf{x}_k)}{P(\mathbf{z}_0, \dots, \mathbf{z}_k)}
$$

9 The probabilistic view of the state estimation is: given a set of measurements $(z_0, ..., z_k)$, can we find a best state x_k to maximize the conditional probabilistic $P(\mathbf{x}_k | \mathbf{z}_1, \dots, \mathbf{z}_k)$?

- How do we describe the uncertainty with Gaussian Noise?
	- > Gaussian distribution

$$
p(x) \sim N(\mu, \sigma^2)
$$

1D (univariate)

$$
p(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1(x-\mu)^2}{2}\sigma^2}
$$

2D+ (multi variate)
$$
p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2} |\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x} - \mu)^t \Sigma^{-1} (\mathbf{x} - \mu)}
$$

 $\mathcal{N}(x|\mu, \sigma^2)$ 2σ $\Sigma_{0.2}$ (X) d -2 $|0$ Opening Minds · Shaping the Future · 啟迪思

How to understand the $P(\mathbf{x}_k | \mathbf{Z}_1, \dots, \mathbf{Z}_k)$

Estimation Formulation

 \mathbf{u}_i : IMU Measurement \mathbf{z}_i : GNSS measurement

States set

 $\chi = {\bf x}_o, {\bf x}_1, {\bf x}_2, \cdots, {\bf x}_k$

Optimal State set
\n
$$
\hat{\chi} = arg \max_{\chi} (P(\chi | \mathbf{Z}, \mathbf{U})) \ P(\chi | \mathbf{Z}, \mathbf{U}) = \prod_{k} P(\mathbf{z}_k | \mathbf{x}_k) \ P(\mathbf{x}_0) \prod_{k} P(\mathbf{x}_k | \mathbf{x}_{k-1}, \mathbf{u}_{k-1})
$$
\n
$$
\text{Bayesian theory}
$$

Kalman Filter

- > In early 1960, American engineer R.E. Kalman discovered a linear minimum variance ESTIMATION method——Kalman Filter. Soon in space technology (such as flying Navigation system, missile guidance, and determination of satellite orbit and attitude) has been applied.
- > Optimal combination of MEASUREMENT and PROPAGATION

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Rudolf Emil Kalman

<http://www.cs.unc.edu/~welch/kalman/media/pdf/Kalman1960.pdf>

A New Approach to Linear Filtering and Prediction Problems¹

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The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:

 (1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinitemamora Gleane

 (2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the ontimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obained without further calculations

(3) The filtering problem is shown to be the dual of the noise-free regulator problem. The new method developed here is applied to two well-known problems, confirming and extending earlier results

The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the Appendix.

Introduction

AN IMPORTANT class of theoretical and practical problems in communication and control is of a statistical nature. Such problems are: (i) Prediction of random signals: (ii) separation of random signals from random noise; (iii) detection of signals of known form (pulses, sinusoids) in the presence of random noice

In his pioneering work. Wiener $[1]^3$ showed that problems (i) and (ii) lead to the so-called Wiener-Hopf integral equation; he also gave a method (spectral factorization) for the solution of this integral equation in the practically important special case of stationary statistics and rational spectra.

Many extensions and generalizations followed Wiener's basic work. Zadeh and Ragazzini solved the finite-memory case [2]. Concurrently and independently of Bode and Shannon [3], they also gave a simplified method [2] of solution. Booton discussed the nonstationary Wiener-Hopf equation [4]. These results are now in standard texts [5-6]. A somewhat different approach along these main lines has been given recently by Darlington [7]. For extensions to sampled signals, see, e.g., Franklin [8], Lees [9]. Another approach based on the eigenfunctions of the Wiener-Hopf equation (which applies also to nonstationary problems whereas the preceding methods in general don't), has been pioneered by Davis [10] and applied by many others, e.g., Shinbrot [11], Blum [12], Pugachev [13], Solodovnikov [14],

In all these works, the objective is to obtain the specification of a linear dynamic system (Wiener filter) which accomplishes the prediction, separation, or detection of a random signal.⁴

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Scientific Research under Contract AF 49 (638)-382. ² 7212 Bellona Ave.

³ Numbers in brackets designate References at end of paper.
⁴ Of course, in general these tasks may be done better by nonlinear filters. At present, however, little or nothing is known about how to obtain (both theoretically and practically) these nonlinear filters

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Present methods for solving the Wiener problem are subject to a number of limitations which seriously curtail their practical u numoce

(1) The optimal filter is specified by its impulse response. It is not a simple task to synthesize the filter from such data.

(2) Numerical determination of the optimal impulse response is often quite involved and poorly suited to machine computation. The situation gets rapidly worse with increasing complexity of

the problem. (3) Important generalizations (e.g., growing-memory filters, nonstationary prediction) require new derivations, frequently of considerable difficulty to the nonspecialist.

(4) The mathematics of the derivations are not transparent. Fundamental assumptions and their consequences tend to be obscured

This paper introduces a new look at this whole assemblage of problems, sidestenning the difficulties just mentioned. The following are the highlights of the paper:

(5) Optimal Estimates and Orthogonal Projections. The Wiener problem is approached from the point of view of conditional distributions and expectations. In this way, basic facts of the Wiener theory are quickly obtained; the scope of the results and the fundamental assumptions appear clearly. It is seen that all statistical calculations and results are based on first and second order averages; no other statistical data are needed. Thus difficulty (4) is eliminated. This method is well known in probability theory (see pp. 75–78 and 148–155 of Doob [15] and pp. 455-464 of Loève [16]) but has not yet been used extensively in engineering.

(6) Models for Random Processes. Following, in particular. Bode and Shannon [3], arbitrary random signals are represented (up to second order average statistical properties) as the output of a linear dynamic system excited by independent or uncorrelated random signals ("white noise"). This is a standard trick in the engineering applications of the Wiener theory [2-7]. The approach taken here differs from the conventional one only in the way in which linear dynamic systems are described. We shall emphasize the concepts of state and state transition; in other words, linear systems will be specified by systems of first-order difference (or differential) equations. This point of view is $=$ (b*(e + 1 - 0 $\frac{9}{2}$ (e) = 1) + π (e) (23) $\mathbf{x}^*(t + s|t) = \hat{E} [\mathbf{x}(t + s)] \mathbf{V}(t)]$

bus Φ^* is also the transition matrix of the linear dynamic system overning the error From (22) we obtain at once a requestor relation for the coariance matrix $\mathbf{P}^*(t)$ of the optimal error $\mathbf{\tilde{x}}(t|t-1)$. Noting that

(*t*) is independent of $\mathbf{x}(t)$ and therefore of $\tilde{\mathbf{x}}(t|t-1)$ we get

 $\mathcal{L}(t+1) = E \widetilde{\mathbf{X}}(t+1|t) \widetilde{\mathbf{X}}(t+1|t).$ $= \mathbf{\Phi}^*(t+1; t) E \widetilde{\mathbf{x}}(t|t-1) \widetilde{\mathbf{x}}'(t|t-1) \mathbf{\Phi}^{*}(t+1; t) + \mathbf{Q}(t)$

 $= \Phi^*(t+1; t)E\widetilde{\mathbf{X}}(t|t-1)\widetilde{\mathbf{X}}(t|t-1)\Phi'(t+1; t) + \mathbf{Q}(t)$

 $= \Phi^*(t+1; t) P^*(t) \Phi^*(t+1; t) + Q(t)$ (24)

here $\mathbf{Q}(t) = E \mathbf{u}(t) \mathbf{u}^{\dagger}(t)$.

* (and thus also for Φ^*). Since,

The estimation error is given by

 $s(n-p)$ co-ordinates.

The expected quadratic loss is

 $\Delta^*(t) = \Phi(t+1; t)P^*(t)W(t)[W(t)P^*(t)W(t)]$

Finally, for any $s \ge 0$, if Φ is invertible

+ $\Phi(t+s; t+1)\Delta^*(t)y(t)$

difference equation is obtained for $\mathsf{P}^*(t)$:

equation in the conventional theory.

recursion relations

 $=\hat{E} [\Phi(t+s; t+1)\mathbf{x}(t+1)]V(t)]$

Theorem 3. (Solution of the Wiener Problem)

 $= \Phi(t + s; t + 1)x^*(t + 10)$ $(s \ge 1)$

If $s \le 0$, the results are similar, but $\mathbf{x}^*(t - s|t)$ will have (1 -

The results of this section may be summarized as follows:

Consider Problem I. The optimal estimate $\mathbf{x}^*(t + 1|t)$ of $\mathbf{x}(t +$

 $x^*(t+1)t = \Phi^*(t+1t) x^*(t+1) + A^*(t) y(t)$ (21)

 (23)

ക

 (27)

 (28)

 (29)

 (30)

 (31)

 $i > i$

 $x + 6x + 10 = 11$

 $t \geq t_0$ (32)

 $^{\circ}$

1) given $y(t_0)$, ..., $y(t)$ is generated by the linear dynamic system

 $cov \widetilde{\mathbf{x}}(t|t-1) = E \widetilde{\mathbf{x}}(t|t-1) \widetilde{\mathbf{x}}'(t|t-1) = \mathbf{P}^*(t)$

 $\sum E \widetilde{x}_i^2(t | t-1) = \text{trace } \mathbf{P}^*(t)$

The matrices $\Delta^*(t)$, $\Phi^*(t + 1; t)$, $\mathbf{P}^*(t)$ are generated by the

In order to carry out the iterations, one must specify the

covariance $P^*(t_0)$ of $x(t_0)$ and the covariance $Q(t)$ of $u(t)$.

= $\Phi(t+s; t+1)\Phi^*(t+1; t)\Phi(t; t+s-1)$

Remarks. (h) Eliminating Δ^* and Φ^* from (28-30), a nonlinear

This equation is linear only if $M(t)$ is invertible but then the

problem is trivial since all components of the random vector $\mathbf{x}(t)$

are observable $\mathbf{P}^*(t+1) = \mathbf{Q}(t)$. Observe that equation (32) plays

a role in the present theory analogous to that of the Wiener-Hopf

Once $\mathbf{P}^*(t)$ has been computed via (32) starting at $t = t_0$, the

explicit specification of the optimal linear filter is immediately

available from formulas (29-30). Of course, the solution of

 (i) The results stated in Theorem 3 do not resolve completely

Problem I. Little has been said, for instance, about the physical

significance of the assumptions needed to obtain equation (25).

the convergence and stability of the nonlinear difference equa-

tion (32), the stability of the optimal filter (21), etc. This can

actually be done in a completely satisfactory way, but must be

left to a future paper. In this connection, the principal guide and

simpler task than solution of the Wiener-Hopf equation.

Equation (32), or of its differential-equation equivalent, is a much

 \times P*($\partial M(t)$) $\Phi'(t+1; t)$ + Q(t)

 $\Phi(t+1; t)\{\mathbf{P}^*(t)-\mathbf{P}^*(t)\mathbf{M}^*(t)[\mathbf{M}(t)\mathbf{P}^*(t)\mathbf{M}^*(t)]^{-1}\}$

There remains the problem of obtaining an explicit formula for $\widetilde{\mathbf{x}}(t+1|t) = \mathbf{\Phi}^*(t+1; t)\widetilde{\mathbf{x}}(t|t-1) + \mathbf{u}(t)$

The covariance matrix of the estimation error is $\widetilde{\mathbf{x}}(t+1)|Z(t)\rangle = \mathbf{x}(t+1) - \widehat{E}[\mathbf{x}(t+1)|Z(t)]$

orthogonal to $\widetilde{\mathbf{y}}$ (t |t - 1), it follows that by (19) that

 $0 = E[\mathbf{x}(t+1) - \mathbf{\Delta}^*(t) \widetilde{\mathbf{y}}(t|t-1)] \widetilde{\mathbf{y}}'(t|t-1)]$ $=E\mathbf{x}(t+1)\widetilde{\mathbf{v}}^{\mathsf{T}}(dt-1)-\mathbf{A}^*(t)E\widetilde{\mathbf{v}}(dt-1)\widetilde{\mathbf{v}}^{\mathsf{T}}(dt-1).$

loting that $\overline{\mathbf{x}}(t+1|t-1)$ is orthogonal to $\overline{Z}(t)$, the definition of

 (17) given earlier, and (17) , it follows further $=$ $E\widetilde{\mathbf{x}}(t+1)t-1\widetilde{\mathbf{v}}(t)t-1)-\mathbf{A}^*(t)\mathbf{M}(t)\mathbf{P}^*(t)\mathbf{M}'(t)$

 $= E[\Phi(t+1)/\tilde{\mathbf{x}}](t-1) + \mathbf{u}(t-1)]\tilde{\mathbf{x}}'(t-1)\mathbf{W}'(t)$

 $\Phi^*(t+1; t) = \Phi(t+1; t) - \Delta^*(t) \mathbf{M}(t)$ $-$ A*COMCOP*COM"CO $P^*(t+1) = \Phi^*(t+1; t)P^*(t)\Phi^*(t+1; t)$

inally, since $H(t)$ is independent of $\mathbf{x}(t)$. $0 = \Phi(r + 1) \cdot \Delta P^*(r) M'(r) = A^*(r) M(r) P^*(r) M'(r)$

low the matrix $M(\triangle P^*(t)M'(t)$ will be positive definite and hence ivertible whenever $P^*(t)$ is positive definite, provided that none f the rows of $M(t)$ are linearly dependent at any time, in other ords, that none of the observed scalar random variables $v(t)$ ${\bf x}^*(t + s|t) = {\bf \Phi}(t + s; t + 1){\bf x}^*(t + 1)|t$ bits, that none of the observed scalar fundom variables $y_1(t)$, ..., $y_2(t)$, ..., $y_3(t)$, ..., ircumstances we get finally

 $\Delta^*(t) = \Phi(t+1; t)P^*(t)W(t)[W(t)P^*(t)W'(t)]^{-1}$ (25)

Since observations start at t_0 , $\widetilde{\mathbf{x}}(t_0|t_0-1) = \mathbf{x}(t_0)$; to begin the erative evaluation of $\mathbf{P}^*(t)$ by means of equation (24), we must so that the estimate $\mathbf{x}^*(t + s|t)$ ($s \ge 0$) is also given by a linear dybyjously specify $P^*(t_*) = Ex(t_*)x'(t_*)$. Assuming this matrix is *namic system of the type* (21). ositive definite, equation (25) then vields $\Delta^*(t_n)$; equation (22) $P^*(t_0 + 1; t_0)$, and equation (24) $P^*(t_0 + 1)$, completing the cycle. now $Q(t)$ is positive definite, then all the $P^*(t)$ will be positive efinite and the requirements in deriving (25) will be satisfied at sch sten

Now we remove the restriction that $t = t + 1$. Since $\mathbf{u}(t)$ is rthogonal to $V(t)$, we have

 $\mathbf{x}^*(t+1|t) = \hat{E} [\mathbf{\Phi}(t+1; t)\mathbf{x}(t) + \mathbf{u}(t)]\mathbf{V}(t)] = \mathbf{\Phi}(t+1; t)\mathbf{x}^*(t|t)$ lence if $\Phi(t+1; t)$ has an inverse $\Phi(t; t+1)$ (which is always the

ase when Φ is the transition matrix of a dynamic system escribable by a differential equation) we have

 $x^*(t|t) = dx(t; t+1)x^*(t+1|t)$

 $t \ge t + 1$, we first observe by repeated application of (16) that $(t+s) = \Phi(t+s; t+1)\mathbf{x}(t+1)$

$$
+\sum_{r=1}^{s-1} \Phi(t+s; t+r) \mathbf{u}(t+r) \qquad (s \ge 1)
$$

ince $\mathbf{u}(t+s-1), \ldots, \mathbf{u}(t+1)$ are all orthogonal to $\mathcal{Y}(t)$,

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the matrices occurring in equation (31) and the covariance matrix of $\tilde{\mathbf{x}}(t|t)$ are found after simple calculations. We have, for all $t \geq 0$

$$
\begin{aligned} \Phi(t; t+1) \Phi^*(t) &= \frac{1}{C_1(t)} \begin{bmatrix} tC_2(t) \\ C_1(t) \\ C_2(t) \end{bmatrix} \\ \mathsf{N}(t; t+1) \Phi^*(t+1; t) \Phi^*(t+1; t) \\ &= \frac{1}{C_1(t)} \begin{bmatrix} C_1(t) - tC_2(t) & C_1(t) - tC_2(t) & -\phi_{21}C_2(t) \\ -C_2(t) & C_1(t) - C_2(t) & -\phi_{31}C_2(t) \\ -C_1(t) + tC_2(t) & -C_1(t) + tC_2(t) & +\phi_{31}C_2(t) \end{bmatrix} \\ \text{and} \\ \text{cov } \widetilde{\mathbf{x}}(t|t) &= E \widetilde{\mathbf{x}}(t|t) \widetilde{\mathbf{x}}(t|t) = -\frac{b^2}{t} \begin{bmatrix} t & t & -t^2 \\ t & 1 & -t \\ t & 1 & -t \end{bmatrix} \end{aligned}
$$

 $\left| \frac{C_1(t)}{C_1(t)} \right|_{t=1}^{t} = \frac{1}{t^2} = \frac{-t}{t^2}$ To gain some insight into the behavior of this system, let us examine the limiting case $t \to \infty$ of a large number of observa-

tions. Then $C_1(t)$ obeys approximately the differential equation $dC_2(t)/dt \approx C_2^2(t)$ $(t \gg 1)$

from which we find

 $C_1(t) \approx (1 - \phi_{13})^2 t^3 / 3 + \phi_{13} (1 - \phi_{13}) t^2 + \phi_{33}^2 t + b^2 / a^2$ $(t \gg 1)$ (39)

Using (39), we get further,

fi i o` $\begin{array}{ccc} 0 & 1 & 0 \end{array}$ and $\Phi^{-1}\Lambda^* \approx$ $(t \gg 1)$ -1 -1 0

Thus as the number of observations becomes large, we depend almost exclusively on $x_1^*(t|t)$ and $x_2^*(t|t)$ to estimate $x_1^*(t+1|t+1)$ 1) and $x^*(t + 1)t + 1$. Current observations are used almost exclusively to estimate the noise

 $x_3^*(t|t) \approx y_1^*(t) - x_1^*(t|t)$ $(t \gg 1)$

One would of course expect something like this since the problem is analogous to fitting a straight line to an increasing number of points

As a second check on the reasonableness of the results given. observe that the case $t \gg 1$ is essentially the same as prediction based on continuous observations. Setting $\phi_{13} = 0$, we have

$$
E \widetilde{x}_1^2(t \,|\, t) \approx \frac{a^2 b^2 t^2}{b^2 + a^2 t^3 / 3} \qquad (t \gg 1; \phi_{33} = 0)
$$

which is identical with the result obtained by Shinbrot [11]. Example 1, and Solodovnikov [14], Example 2, in their treatment of the Wiener problem in the finite-length, continuous-data case, using an approach entirely different from ours.

Conclusions

This paper formulates and solves the Wiener problem from the "state" point of view. On the one hand, this leads to a very general treatment including cases which cause difficulties when attacked by other methods. On the other hand, the Wiener problem is shown to be closely connected with other problems in the theory of control. Much remains to be done to exploit these connections

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Kalman Filter——Navigation application using dead reckoning and visual measurement to landmark

Examples of Kalman Filter in Navigation

Definitions

- $>$ X, The *state vector* is the set of parameters describing a system, known as *states*, which the Kalman filter estimates.
- > P, Associated with the state vector is an *error covariance matrix*. This represents the uncertainties in the Kalman filter's state estimates and the degree of correlation between the errors in those estimates.

North

Definitions

- > F, The *system model*, also known as the process model or time-propagation model, describes how the Kalman filter states and error covariance matrix vary with time. (The system model is deterministic for the states as it is based on known properties of the system.)
- $> Q$, A state uncertainty should also be increased with time to account for unknown changes in the system that cause the state estimate to go out of date in the absence of new measurement information. This variation in the true values of the states is known as *system noise* or *process noise*.

Definitions

- > Z, The *measurement vector* is a set of simultaneous measurements of properties of the system which are functions of the state vector.
- $>$ R, Associated with the measurement vector is a *measurement noise covariance* matrix which describes the statistics of the noise on the measurements.
- > **H, Z=H(X)=HX**, The *measurement model* describes how the measurement vector varies as a function of the true state vector (as opposed to the state vector estimate) in the absence of measurement noise.

 $>\Delta Z_t = Z_t - H X_t^ -S_t = HP_{t-1}^-H^T + R$ $>$ K_t = P_{t-1}^- H^TS_t⁻¹ $>\mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t \Delta \mathbf{Z}_t$ $> P_t^+ = (I - K_t H) P_{t-1}^-$ − 20

Measurement Update

 $> X_t^-$ = $FX_{t-1}^+ + BU_t$ $> P_t^- = FP_{t-1}^+F^T + Q$

Propagation

Key Equations of Kalman Filter

Measurement Innovation, ∆

$$
>\Delta Z_t = Z_t - H X_t^-
$$

- >Meaning: The difference between propagation and measurement in the domain of Z (measurement)
- $\Delta Z_t = 0$, What does it mean?
- $\Delta Z_t \neq 0$, What shall we do?

Kalman Filter Gain,

$$
\begin{aligned} &> \mathbf{K}_t = \mathbf{P}_{t-1}^-\mathbf{H}^{\mathrm{T}}\mathbf{S}_t^{-1} \\ &> \mathbf{S}_t = \mathbf{H}\mathbf{P}_{t-1}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R} \end{aligned}
$$

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HX
Domain (Space)
Transaction
$\mathbf{Z}^T = \mathbf{X}^T \mathbf{H}^T$

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If we forget there is no denominator in Matrix…

The updated state covariance (uncertainty) considering (+) propagation covariance (uncertainty)

The propagated state covariance (uncertainty) in the domain of the transpose of **Z** (measurement)

=

The propagated state covariance (uncertainty) in the domain of **Z** (measurement) + measurement covariance (uncertainty) **R**

Kalman Filter Gain,

- $>K_t =$ $\mathbf{FP}_{t-1}^+ \mathbf{F}^\mathrm{T} + \mathbf{Q} \big) \mathbf{H}^\mathrm{T}$ $HP_{t-1}^-H^T+R$
- $>$ **R** $>$ **Q**, what does it mean?
- $\mathbf{R} \ll \mathbf{Q}$, what does it mean?
- $> R = Q$, what does it mean?

Simplified the Models to Identity Matrix, I
$$
f = I
$$

\n
$$
>X_t^- = FX_{t-1}^+ + BU_t
$$
\n
$$
>Y_t = FP_{t-1}^+H^T + Q
$$
\n
$$
>X_t = Z_t - HX_t^-
$$
\n
$$
>X_t = P_{t-1}^-H^T + R
$$
\n
$$
>K_t = P_{t-1}^-H^TS_t^{-1}
$$
\n
$$
>X_t^+ = X_t^- + K_t\Delta Z_t
$$
\n
$$
>Y_t^+ = (I - K_tH)P_{t-1}
$$
\n
$$
>Y_t^+ = (I - K_t)P_{t-1}^{-1}
$$

×.

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What Kalman Filter tries to achieve?

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Fusion of Gaussian distribution in 1D

After some calculation and rearrangements Distribution of two measurements $\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} (q_2 - q_1)$ $\hat{q}_1 = q_1$ with variance σ_1^2 Kalman Gain $\hat{q}_2 = q_2$ with variance σ_2^2 $f(q)$ $f(q) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{(q-\mu)^2}{2\sigma^2}\right)$ > Weighted least-square $S = \sum_{i=1}^{n} w_i (\hat{q} - q_i)^2$ $i = 1$ If $w_i = 1/\sigma_i^2$ > Finding minimum error $\frac{\partial S}{\partial \hat{q}} = \frac{\partial}{\partial \hat{q}} \sum_{i=1}^{n} w_i (\hat{q} - q_i)^2 = 2 \sum_{i=1}^{n} w_i (\hat{q} - q_i) = 0$ $q₂$ q_1

Question: Why Kalman Filter is Optimal?

- >What does it mean to Optimal?
	- Maximum the Probability of the Estimation!
- >What assumptions are made in Kalman Filter so that it can achieve Optimal?
	- A. Gaussian Random Variable (Gaussian Noise)
	- B. 1st Order of Markov Chain

Kalman filter in GNSS

>**Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter**

>Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter

Statistical Methods for Estimation and Optimization

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Batch (including all the data in the past) Optimization

Batch (including all the data in the past) Optimization

Batch (including all the data in the past) Optimization

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- prediction prior.
- Which one do you trust more, your prior or your measurement ?

Kalman filter in GNSS

>Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter

>**Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter**

Evaluation of GNSS Positioning

GNSS positioning performance using the three listed methods

Accommunical and Aviation Engineering 香港理工大學 WLS* : weighted least square with pseudorange EKF* : Pseudorange/Doppler fusion with extended Kalman filter

lisciplinary Division of

FGO* : Pseudorange/Doppler fusion with factor graph optimization

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pseudorange Huawei P40 Pro Phone

航空工程跨領域學部 香港理工大學 WLS^{*}: weighted least square with EKF* : Pseudorange/Doppler fusion with extended Kalman filter FGO* : Pseudorange/Doppler fusion with factor graph optimization

**Interdisciplinary Division of
Aeronautical and Aviation Engineering**

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Evaluation with Huawei P40 Pro

Q&A

Thank you for your attention \odot Q&A

Dr. Weisong Wen

If you have any questions or inquiries, please feel free to contact me.

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Supplementary: GNSS/INS Integration Using Kalman Filtering

Service Service

Inertial navigation system

 \hat{a}_t , $\hat{\omega}_t$ are the raw accelerometer and gyroscope measurements in the body frame a_t , ω_t are expected measurements

The cap \land denotes the noisy measurement or estimation of a certain quantity

$$
\hat{\mathbf{a}}_t = \mathbf{a}_t + \mathbf{R}_w^t \mathbf{g}^w + \mathbf{b}_{a_t} + \mathbf{n}_a \tag{1}
$$

 $\hat{\mathbf{\omega}}_t = \mathbf{\omega}_t + \mathbf{b}_{\omega_t} + \mathbf{n}_{\omega} (2)$

 $\mathbf{n}_a \sim \mathcal{N}(0, \sigma_a^2), \, \mathbf{n}_\omega \sim \mathcal{N}(0, \sigma_\omega^2)$

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Error analysis of inertial navigation system

- > The errors of **accelerometer and gyroscope** can be divided into: **deterministic error & random error.**
- > Deterministic errors can be calibrated in advance including bias, scale...
- > Random error usually assumes that noise obeys Gaussian distribution, including Gaussian white noise, bias random walk...

Deterministic error

(sourcing imperfectness of electrical/mechanical components)

> Bias: In theory, the output of the IMU sensor should be 0 when there is no external action. However, there is a bias **b** to the international data. Influence of accelerometer bias on orientation estimation:

$$
V_{error} = \mathbf{b}_{a}t, \mathbf{P}_{error} = \frac{1}{2}\mathbf{b}_{a}t^{2}
$$

> Scale: The ratio between the actual value and the sensor output value.

> Nonorthogonality/Misalignment Errors: When manufacturing multi-axis IMU sensors, due to the manufacturing process, the xyz axis may not be vertical.

Deterministic error calibration method—Accelerometer

> The six-sided method means that the three axes of the accelerometer are placed horizontally up or down for a period of time, and data on the six sides are collected to complete the calibration.

If the axes are orthogonal, it is easy to get bias and scale:

$$
b_{a} = \frac{l_{f}^{up} + l_{f}^{down}}{2}
$$

$$
s_{a} = \frac{l_{f}^{up} - l_{f}^{down}}{2 \cdot g} = \begin{bmatrix} s_{a,xx} \\ s_{a,yy} \\ s_{a,zz} \end{bmatrix}
$$

$$
l_{f}^{down}
$$

I is the measured value of a certain axis of the accelerometer, g is the local gravity acceleration

Deterministic error calibration method—Accelerometer

> When considering the inter-axis error, the relationship between the actual acceleration and the measured value is:

$$
\begin{bmatrix} l_{ax} \\ l_{ay} \\ l_{az} \end{bmatrix} = \begin{bmatrix} s_{xx} & m_{xy} & m_{xz} \\ m_{yz} & s_{yy} & m_{yz} \\ m_{zx} & m_{zy} & s_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} b_{ax} \\ b_{ay} \\ b_{az} \end{bmatrix}
$$

> When placed horizontally and statically on 6 sides, the theoretical value of acceleration is

$$
\mathbf{a_1} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \mathbf{a_4} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}, \mathbf{a_5} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}, \mathbf{a_6} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}
$$

> Corresponding measurement value matrix **L**

$$
\mathbf{L} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \end{bmatrix}
$$

> 12 variables can be obtained by using least squares.

Deterministic error calibration method——Gyroscope

> Unlike the six-sided method of accelerometer, the true value of the gyroscope is provided by a high-precision turntable. The 6 faces in this refer to the clockwise and counterclockwise rotation of each axis

high-precision three-axis turntable

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Random error – Unstableness of electrical and mechanical component due to temperature

- > We can calibrate to do temperature compensation on the bias and scale estimated by the sensor, and to obtain the values of bias and scale at different temperatures and draw them into a curve.
- > Soak method: control the temperature value of **the constant temperature room**, and then read the sensor value for calibration.

The thin solid lines are the results under separated heating and cooling processes The thick lines are the final curve fitted result .

- $>$ ω_t^{Gyro} : measured 3-axis rotation by the gyroscopes at epoch *t*
- $> X_{INS,t}^{body}$:estimated 3-axis position in body frame by INS at epoch *t*
- $> V_{INS,t}^{body}$: estimated 3-axis velocity in body frame by INS at epoch *t*
- $> \Psi_{INS,t}^{body}$: estimated 3-axis orientation in body frame (Euler angles) by INS at epoch *t*
- > B^{*body*}:estimated 3–axis biaes of accelerometers in body frame at epoch *t*
- > B^{*body*}:estimated 3–axis biaes of gyroscopes in body frame at epoch *t*
- $>$ W_{b_a} : estimated 3-axis random walk noise of accelerometers in body frame
- $>$ W_{b_a} : estimated 3-axis random walk noise of gyroscopes in body frame

INS Mechanization

Sola, Joan. "Quaternion kinematics for the error-state Kalman filter." *arXiv preprint arXiv:1711.02508* (2017).

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The Euler angle rates obtained by angular velocity:

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\Psi}_{INS,t}^{body} = \begin{bmatrix} 1 & sin(\phi) \tan(\theta) & cos(\phi) \tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) \sec(\theta) & cos(\phi) \sec(\theta) \end{bmatrix} \omega_t^{Gyro} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}
$$

$$
\delta \Psi_{INS,t}^{body} = \dot{\Psi}_{INS,t}^{body} \Delta t = \begin{bmatrix} \delta \phi \\ \delta \theta \\ \delta \psi \end{bmatrix}
$$

Rotate the Euler angles from Body to Local

$$
\Psi_{INS,t}^{local} = \mathbf{R}_{body}^{local} \Psi_{INS,t}^{body}
$$

Update the current Euler angles

 $\Psi_{INS,t}^{body} = \Psi_{INS,t-1}^{body} + \delta \Psi_{INS,t}^{body}$

$$
\mathbf{R}_{body}^{local} = \mathbf{R}(X, \phi)\mathbf{R}(Y, \theta)\mathbf{R}(Z, \psi)
$$

=
$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

INS Mechanization

To remove the gravity from the acceleration

$$
\mathbf{a}_{INS,t}^{local} = \mathbf{R}_{body}^{local} \mathbf{a}_{INS,t}^{body} - \mathbf{g}
$$

To obtain the change of aircraft in terms of position, velocity and orientation

$$
\delta \mathbf{X}_{INS,t}^{local} = \mathbf{X}_{INS,t}^{local} - \mathbf{X}_{INS,t-1}^{local} = \mathbf{V}_{INS,t-1}^{local} \Delta t + \frac{\mathbf{a}_{INS,t}^{local} \Delta t^2}{2}
$$

 $\boldsymbol{\delta} \mathbf{V}^{local}_{INS,t} = \mathbf{V}^{local}_{INS,t} - \mathbf{V}^{local}_{INS,t-1} = \boldsymbol{a}^{local}_{INS,t} \Delta t$

$$
\delta\Psi_{INS,t}^{local} = \Psi_{INS,t}^{local} - \Psi_{INS,t-1}^{local}
$$

Kalman Filter——GNSS/INS(Open Loop)

System States:

 $\mathbf{X}_t = \left(\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \mathbf{W}_t^{local} \right)$ \mathbf{X}_{t}^{local} = $\left(x_{t}^{local}, y_{t}^{local}, z_{t}^{local}\right)$ $\mathbf{V}_{t}^{local} = \left(v x_{t}^{local}, v y_{t}^{local}, v z_{t}^{local}\right)$ Ψ_t^{local} = $\left(\phi_{roll}, \theta_{pitch}, \psi_{yaw}\right)$

Propagation model:

 $X_t^- = FX_{t-1}^+ + BU_t$ $\mathbf{F} =$ 1 ⋯ 0 \vdots \ddots \vdots 0 ⋯ 1 $B =$ 1 ⋯ 0 \vdots \ddots \vdots 0 ⋯ 1 $\mathbf{U}_t =$ $\delta{\rm X}^{local}_{INS,t}$ $\delta V_{INS,t}^{local}$ $\delta \Psi _{INS,t}^{local}$ Measurement model:

$$
\Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-
$$

$$
H = \begin{bmatrix} I_{3\times 3} & \cdots & 0 \\ \vdots & & 0 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}
$$

$$
\mathbf{Z}_{t} = \mathbf{X}_{GNSS,t}^{local} = \begin{bmatrix} x_{GNSS,t}^{local} \\ y_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \end{bmatrix}
$$

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Closed-loop Correction

- > The estimated position, velocity, and attitude errors are fed back to the inertial navigation processor, where they are used to correct the inertial navigation solution itself.
- > any accelerometer and gyro errors estimated by the Kalman filter are fed back to correct the IMU measurements, as they are input to the inertial navigation equations.
- > Unlike the position, velocity, and attitude corrections, the accelerometer and gyro corrections must be applied on every iteration

INS Mechanization

The Euler angle rates obtained by angular velocity:

$$
\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\Psi}_{INS,t}^{body} = \begin{bmatrix} 1 & sin(\phi) \tan(\theta) & cos(\phi) \tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) \sec(\theta) & cos(\phi) \sec(\theta) \end{bmatrix} (\omega_t^{Gyro} - B_{\omega,t-1}^{body})
$$

$$
\delta \Psi_{INS,t}^{body} = \dot{\Psi}_{INS,t}^{body} \Delta t = \begin{bmatrix} \delta \phi \\ \delta \theta \\ \delta \psi \end{bmatrix}
$$

Update the current Euler angles

$$
\Psi_{INS,t}^{body} = \Psi_{KF,t-1}^{body} + \delta \Psi_{INS,t}^{body}
$$

$$
\Psi_{KF,t-1}^{local} = \mathbf{R}_{body}^{local} \Psi_{KF,t-1}^{body}
$$

Rotate the Euler angles from Body to Local

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$$
\Psi_{INS,t}^{local} = \mathbf{R}_{body}^{local} \Psi_{INS,t}^{body}
$$

$$
\mathbf{R}_{body}^{local} = \mathbf{R}(X, \phi)\mathbf{R}(Y, \theta)\mathbf{R}(Z, \psi)
$$

=
$$
\begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

INS Mechanization

To remove the gravity from the acceleration

 $\mathbf{a}^{local}_{INS,t} = \mathbf{R}^{local}_{body} \big(\mathbf{a}^{body}_{INS,t} - \mathbf{B}^{body}_{a,t-1} \big) - \mathbf{g}$

To obtain the change of aircraft in terms of position, velocity and orientation

$$
\delta \mathbf{X}_{INS,t}^{body} = \mathbf{V}_{INS,t-1}^{local} \Delta t + \frac{\mathbf{a}_{INS,t}^{local} \Delta t^2}{2}
$$

 $\boldsymbol{\delta} \mathbf{V}^{local}_{INS,t} = \boldsymbol{a}^{local}_{INS,t} \Delta t$

 $\boldsymbol{\delta}\mathbf{\Psi}_{INS,t}^{local} = \mathbf{\Psi}_{KF,t-1}^{local} - \mathbf{\Psi}_{INS,t-1}^{local}$

 $\boldsymbol{\delta} \mathbf{X}^{local}_{INS,t}$

Kalman Filter——GNSS/INS (Closed Loop)

System States:

$$
\mathbf{X}_{t} = (\mathbf{X}_{t}^{local}, \mathbf{V}_{t}^{local}, \mathbf{\Psi}_{t}^{local}, \mathbf{B}_{a,t}^{body}, \mathbf{B}_{\omega,t}^{body})
$$
\n
$$
\mathbf{X}_{t}^{local} = (x_{t}^{local}, y_{t}^{local}, z_{t}^{local})
$$
\n
$$
\mathbf{V}_{t}^{local} = (vx_{t}^{local}, vy_{t}^{local}, vz_{t}^{local})
$$
\n
$$
\mathbf{\Psi}_{t}^{local} = (\phi_{roll}, \theta_{pitch}, \psi_{yaw})
$$
\n
$$
\mathbf{B}_{a,t}^{body} = (b_{ax,t}^{body}, b_{ay,t}^{body}, b_{ady}^{body})
$$
\n
$$
\mathbf{B}_{\omega,t}^{body} = (b_{\omega x,t}^{body}, b_{\omega y,t}^{body}, b_{\omega z,t}^{body})
$$

Propagation model:

$$
\mathbf{X}_t^- = \mathbf{F} \mathbf{X}_{t-1}^+ + \mathbf{B} \mathbf{U}_t
$$
\n
$$
\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \mathbf{U}_t = \begin{bmatrix} \delta \mathbf{V}_{INS,t}^{local} \\ \delta \mathbf{V}_{INS,t}^{local} \\ W_{b_{\omega}} \\ W_{b_{\omega}} \end{bmatrix}
$$

Measurement model:

$$
\Delta \mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H} \mathbf{X}_t^-
$$

$$
\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3 \times 3} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}
$$

$$
\mathbf{Z}_{t} = \mathbf{X}_{GNSS,t}^{local} = \begin{bmatrix} x_{GNSS,t}^{local} \\ y_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \end{bmatrix}
$$

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Closed Loop

>Chapters 3 and 5, Paul D. Groves, *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems, 2nd Edition*, Artech House, 2013.