# State Estimation: Factor Graph for Integrated Navigation I

## AAE4203 – Guidance and Navigation

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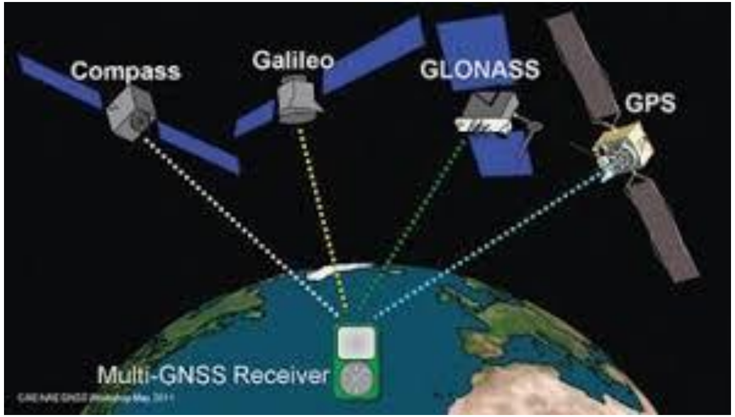
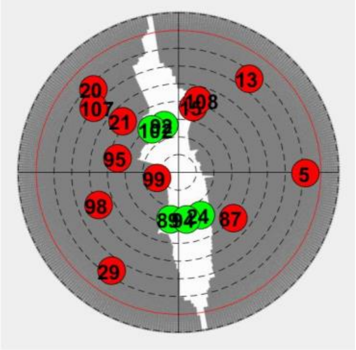
# GNSS Benefits

> GNSS provides a high long-term position accuracy with errors limited to a few meters (stand-alone), while user equipment is available for less than $100 (€80).

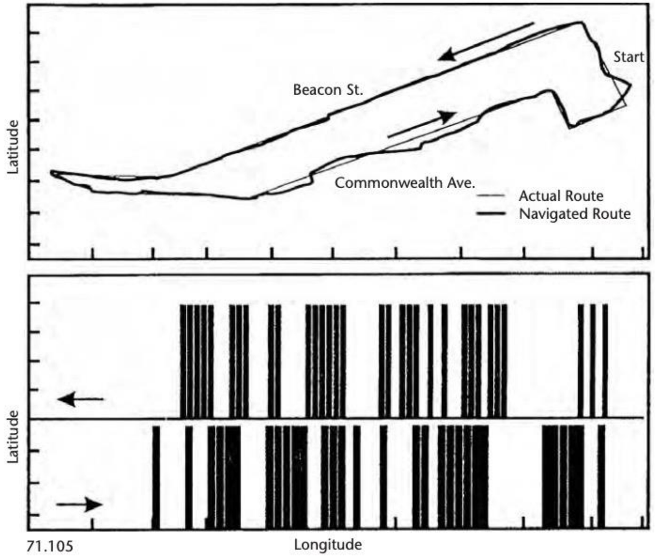


# GPS Drawbacks

> One primary concern with using GPS as a stand-alone source for navigation is **signal interruption**.

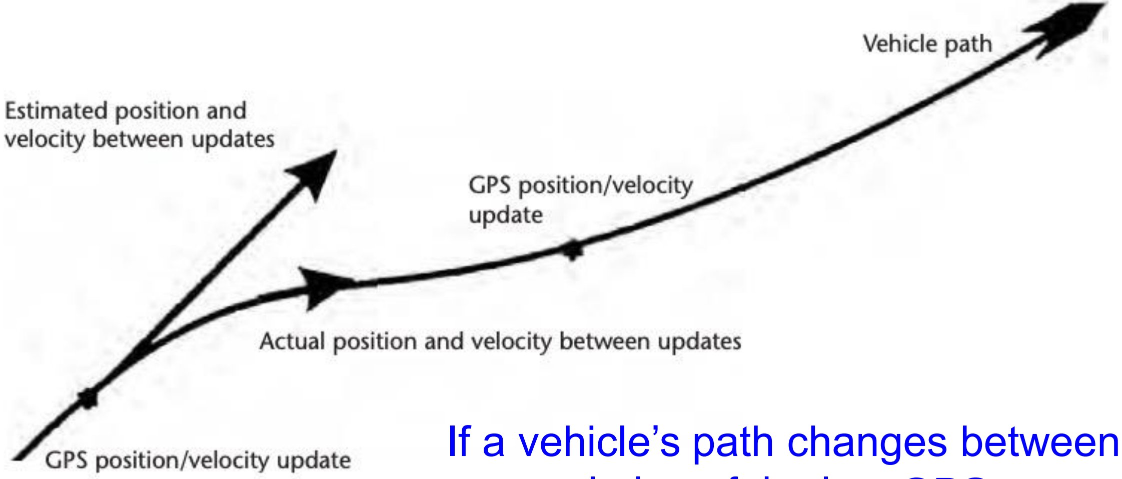



If the number of usable satellites is less than three, some receivers have the option of not producing a solution or extrapolating the last position and velocity solution forward in what is called *dead-reckoning* (DR) navigation. Inertial navigation systems (INSs) can be used as a flywheel to provide navigation during shading outages.



# GPS Drawbacks

> The **low update rate** of the GPS observations in some equipment is also of concern in real-time applications, especially those related to vehicle control.




If a vehicle's path changes between updates, the extrapolation of the last GPS measurement produces an error in the estimated and true position.

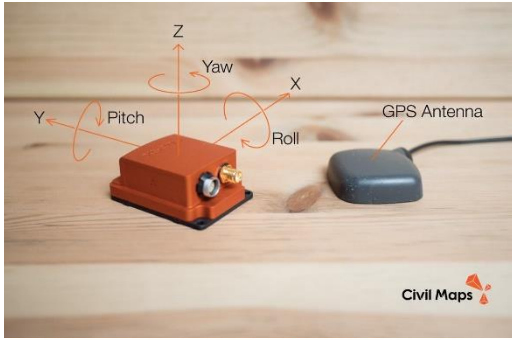
# Framework of Sensors to Navigation

| | **Sensing** | **Measuring** | **Measurement** | **Model** | **Estimate** |
|---|---|---|---|---|---|
| *GNSS* | GNSS Signal → | Signal Defined Receiver → | Pseudorange → | Least Square Estimation → | Point Position |
| *INS* | Raw measurements (voltage, delays, etc) → | Accelerometer/ Gyroscope Models → | Acceleration & Angular Velocity → | INS Mechanization → | Relative Position |
| *Visual Navigation* | Images → | Features Detection & Tracking → | Associated Features → | Least Square Estimation → | Relative Position |

# Sensor Integration Based on Kalman Filter

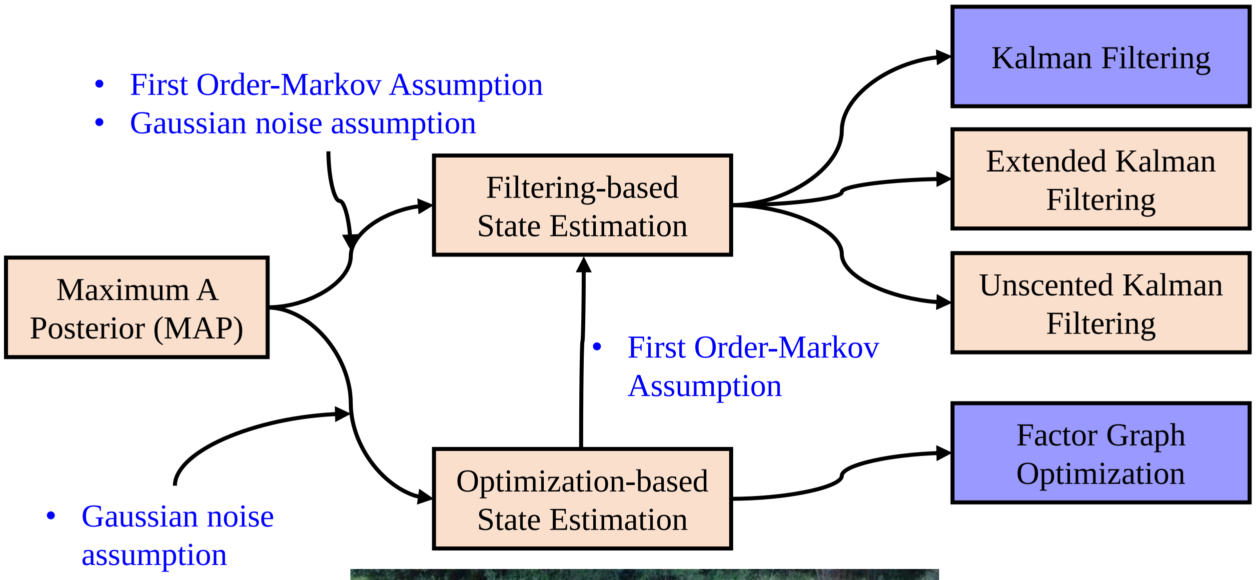
# How the Sensor Fusion Problem Looks like…

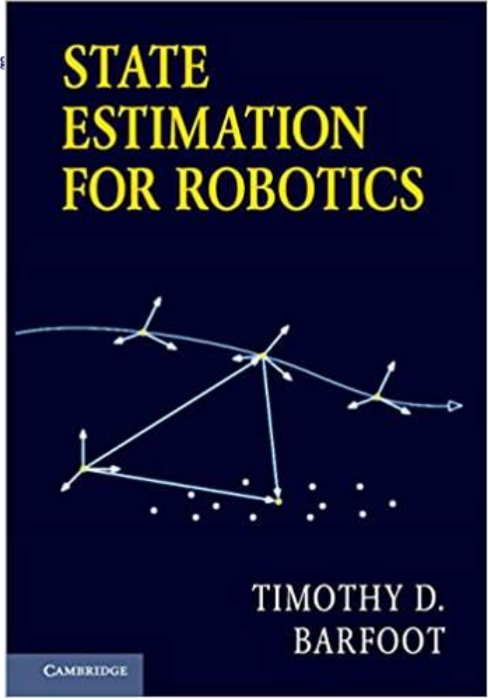
- GPS provide the position in $x, y, z$
- IMU provide the linear angular velocity in $x, y, z$
- GPS Doppler provide velocity in $x, y, z$
- Visual positioning provide relative motion $\Delta x, \Delta y, \Delta z$




How to achieve this sensor fusion by combining the positioning from different sources?

# State Estimation Methods



# Basics for Probabilistic

Event A and B. $P(\mathrm{A})$ denotes the probability that the event A happens.

$$P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{AB})}{P(\boldsymbol{A})} \Rightarrow P(\mathrm{AB}) = P(\mathrm{B}|\mathrm{A})P(\boldsymbol{A}) \Rightarrow P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{B}|\mathrm{A})P(\boldsymbol{A})}{P(\boldsymbol{A})}$$

Event z denotes the measurements. $P(\mathrm{z})$ denotes the probability that the event A happens.

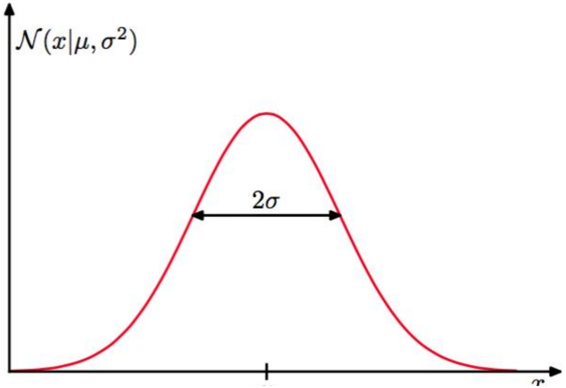
$$P(\mathbf{x}_k|\mathbf{z}_1, \ \ldots, \ \mathbf{z}_k) = \frac{P(\mathbf{z}_0, \ \ldots, \ \mathbf{z}_k|\mathbf{x}_k)P(\mathbf{x}_k)}{P(\mathbf{z}_0, \ \ldots, \ \mathbf{z}_k)}$$

The probabilistic view of the state estimation is: given a set of measurements $(\mathbf{z}_0, \ \ldots, \ \mathbf{z}_k)$, can we find a best state $\mathbf{x}_k$ to maximize the conditional probabilistic $P(\mathbf{x}_k|\mathbf{z}_1, \ \ldots, \ \mathbf{z}_k)$?

# How do we describe the uncertainty with Gaussian Noise?

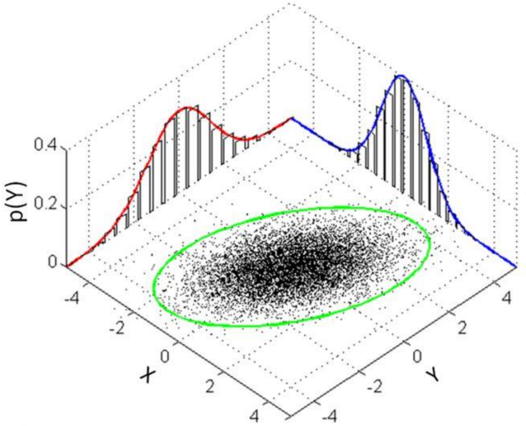
> Gaussian distribution

$$p(x) \sim N(\mu, \sigma^2)$$




1D (univariate)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$




2D+ (multi variate)

$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

# How to understand the $P(\mathbf{x}_k|\mathbf{z}_1, \ \ldots, \ \mathbf{z}_k)$

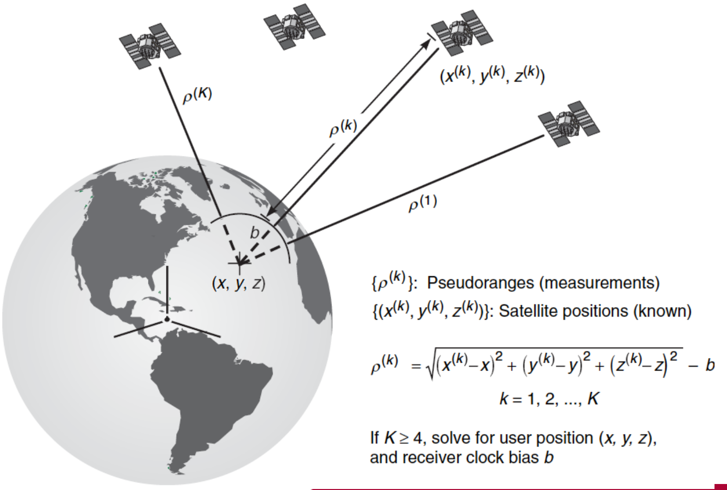
Observation function for pseudorange (code) measurement

$$\rho_{r,t}^{s} = r_{r,t}^{s} + c(\delta_{r,t} - \delta_{r,t}^{s}) + I_{r,t}^{s} + T_{r,t}^{s} + \varepsilon_{r,t}^{s}$$

- $\rho_{r,t}^{s}$: Pseudorange
- $r_{r,t}^{s}$: Range distance → $||\mathbf{p}_{t}^{G,s} - \mathbf{p}_{r,t}^{G}||$
- $\delta_{r,t}$: Receiver clock Bias (1~2m)
- $\delta_{r,t}^{s}$: Satellite clock bias
- $I_{r,t}^{s}$: ionospheric delay Distance (1~2m)
- $T_{r,t}^{s}$: tropospheric delay Distance (1~2m)
- $\varepsilon_{r,t}^{s}$: multipath effects, NLOS receptions, receiver noise, antenna phase-related noise (0~100m)

$\rho_{r,t}^{s}$ → $z_{r,t}^{s}$ $\qquad$ $\mathbf{p}_{r,t}^{G}$ → $\mathbf{x}_{r,t}^{G}$

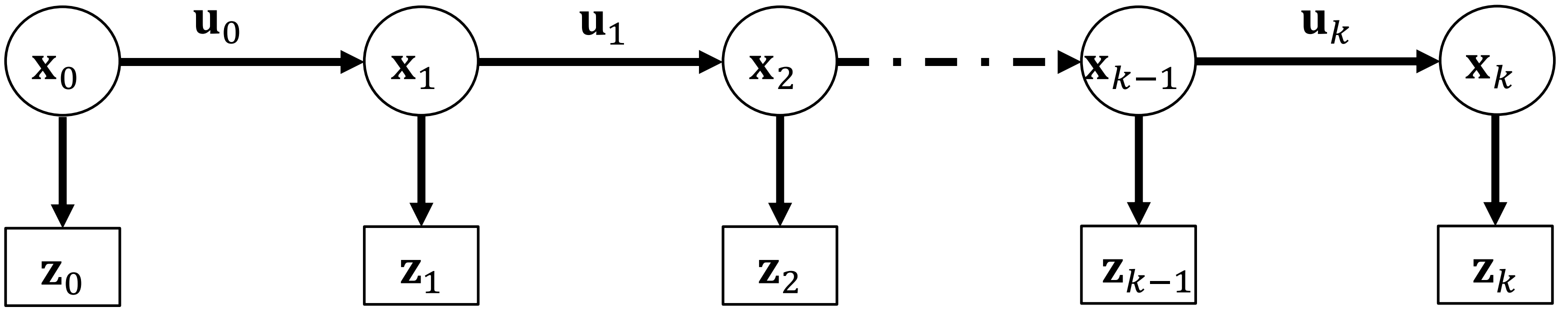
$$P(\mathbf{x}_k|\mathbf{z}_1, \ \ldots, \ \mathbf{z}_k) = \frac{P(\mathbf{z}_0, \ \ldots, \ \mathbf{z}_k|\mathbf{x}_k)P(\mathbf{x}_k)}{P(\mathbf{z}_0, \ \ldots, \ \mathbf{z}_k)}$$




**Formulate the $P(\mathbf{x}_k|\mathbf{z}_1, \ \ldots, \ \mathbf{z}_k)$ for the GNSS pseudorange measurements!**

# Estimation Formulation

$\mathbf{u}_i$: IMU Measurement
$\mathbf{z}_i$: GNSS measurement




**States set**

$$\chi = \{\mathbf{x}_o, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$$

The **maximum a posteriori (MAP)** estimate is given by

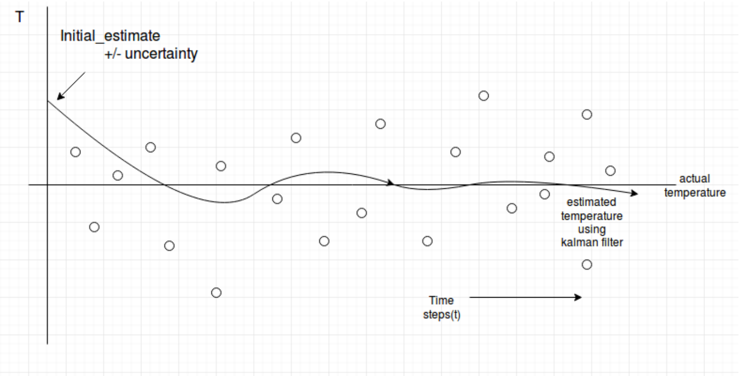
**Optimal State set**

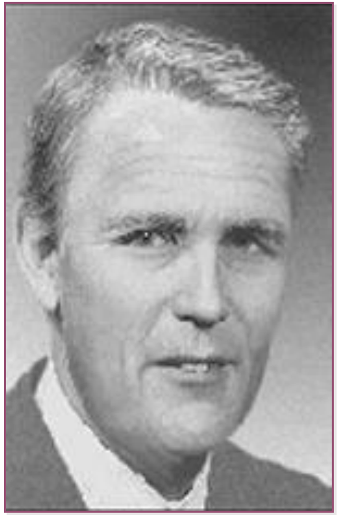
$$\hat{\chi} = arg \max_{\chi}(P(\chi|\mathbf{Z}, \mathbf{U})) \quad P(\chi|\mathbf{Z}, \mathbf{U}) = \prod_k P(\mathbf{z}_k|\mathbf{x}_k) \, P(\mathbf{x}_0) \prod_k P(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{u}_{k-1})$$

Bayesian theory

# Kalman Filter

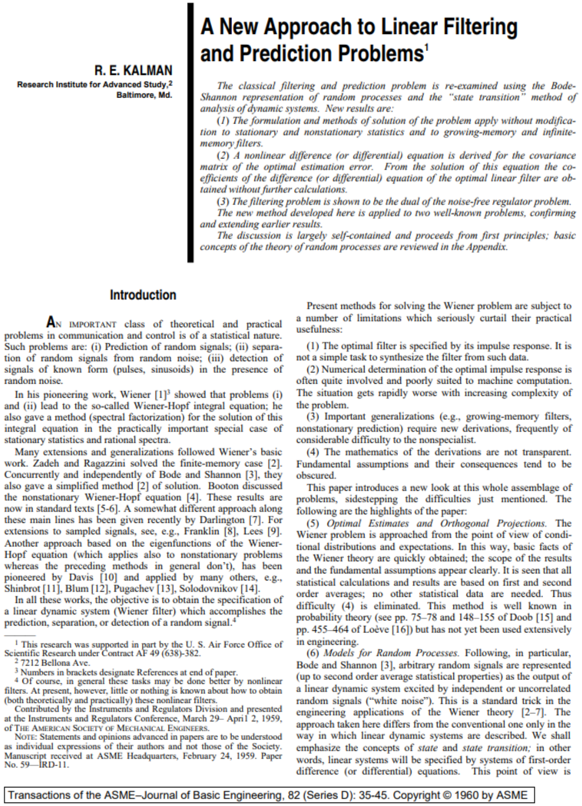
- In early 1960, American engineer R.E. Kalman discovered a linear minimum variance ESTIMATION method——Kalman Filter. Soon in space technology (such as flying Navigation system, missile guidance, and determination of satellite orbit and attitude) has been applied.
- Optimal combination of MEASUREMENT and PROPAGATION



Rudolf Emil Kalman

http://www.cs.unc.edu/~welch/kalman/media/pdf/Kalman1960.pdf



# A New Approach to Linear Filtering and Prediction Problems[1]

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Research Institute for Advanced Study,[2]
Baltimore, Md.

*The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the "state transition" method of analysis of dynamic systems. New results are:*

*(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-memory filters.*

*(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the coefficients of the difference (or differential) equation of the optimal linear filter are obtained without further calculations.*

*(3) The filtering problem is shown to be the dual of the noise-free regulator problem.*

*The new method developed here is applied to two well-known problems, confirming and extending earlier results.*

*The discussion is largely self-contained and proceeds from first principles; basic concepts of the theory of random processes are reviewed in the Appendix.*

## Introduction

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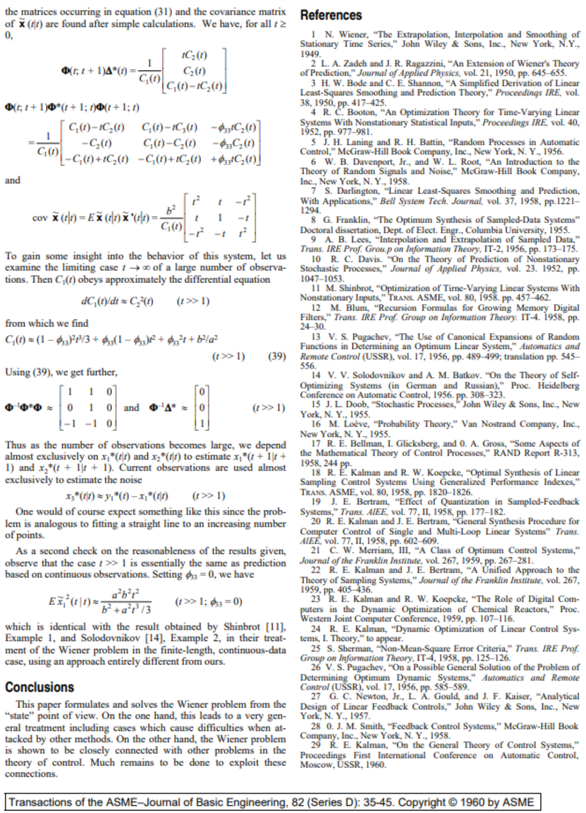

## Conclusions

## References

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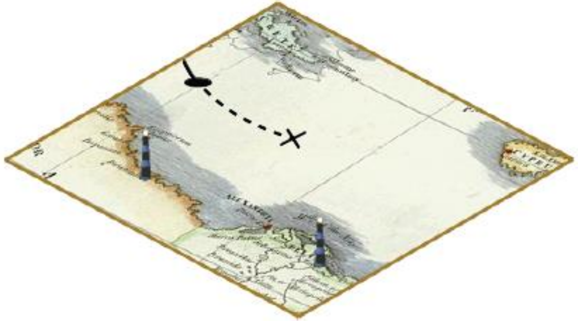
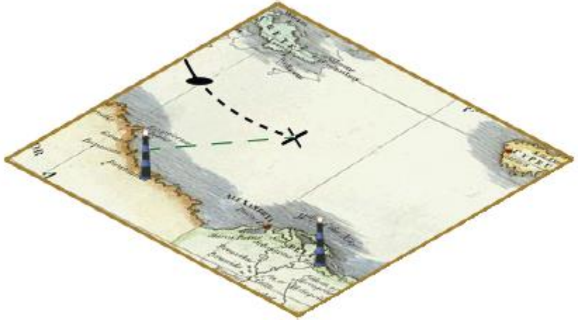
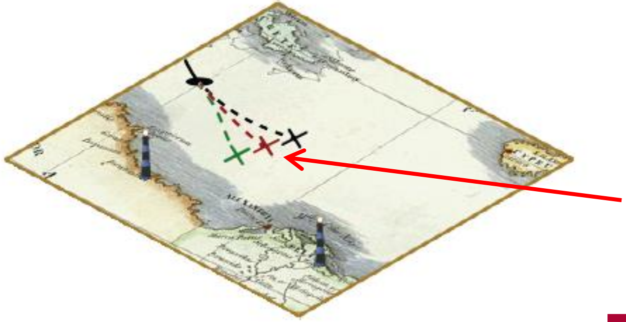
# Kalman Filter——Navigation application using dead reckoning and visual measurement to landmark

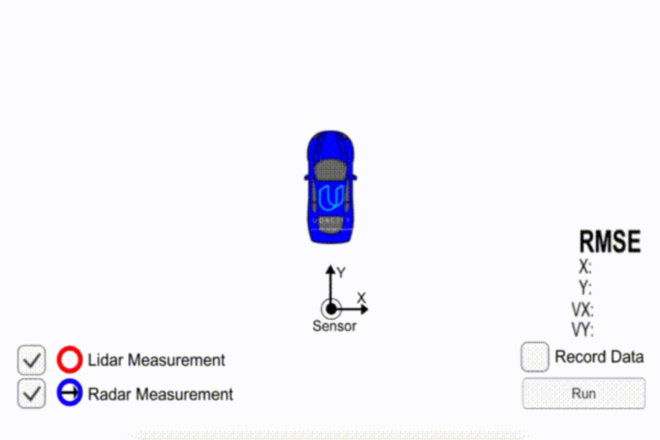


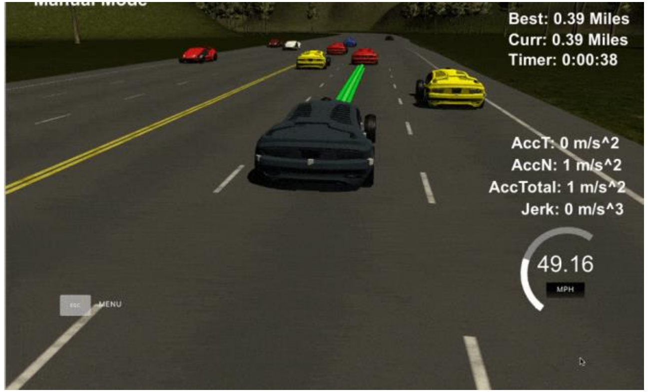


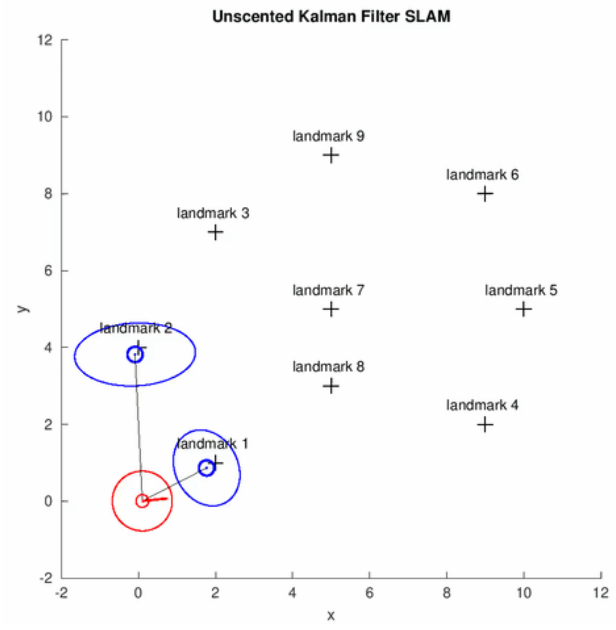


**Predicted and corrected position of the ship**

# Examples of Kalman Filter in Navigation



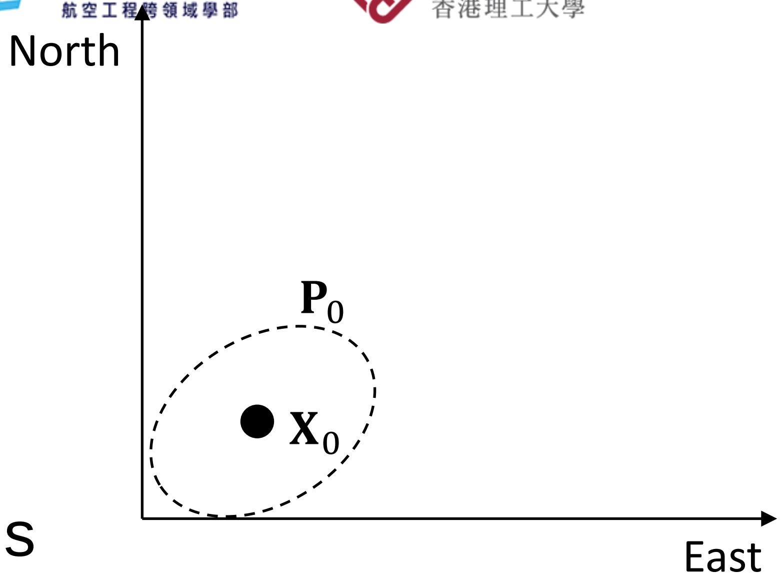



# Definitions

> **X**, The *state vector* is the set of parameters describing a system, known as *states*, which the Kalman filter estimates.

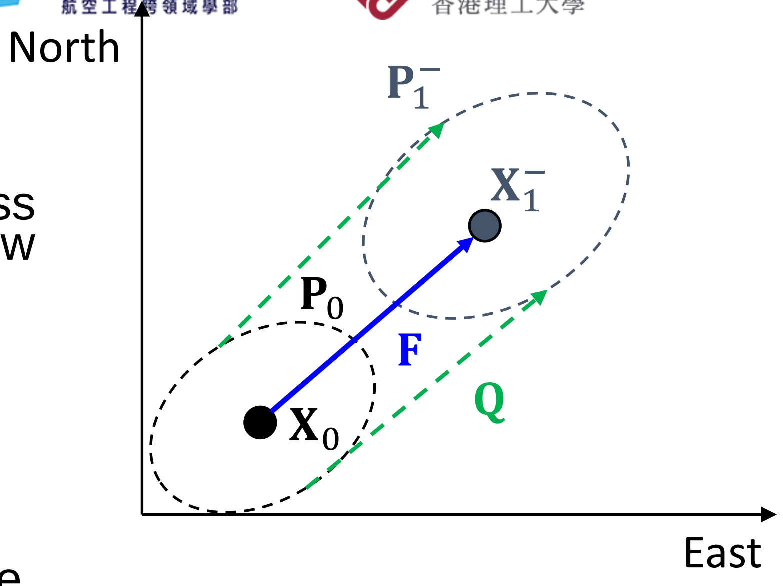
> **P**, Associated with the state vector is an *error covariance matrix*. This represents the uncertainties in the Kalman filter's state estimates and the degree of correlation between the errors in those estimates.



# Definitions

- **F**, The *system model*, also known as the process model or time-propagation model, describes how the Kalman filter states and error covariance matrix vary with time. (The system model is deterministic for the states as it is based on known properties of the system.)

- **Q**, A state uncertainty should also be increased with time to account for unknown changes in the system that cause the state estimate to go out of date in the absence of new measurement information. This variation in the true values of the states is known as *system noise* or *process noise*.

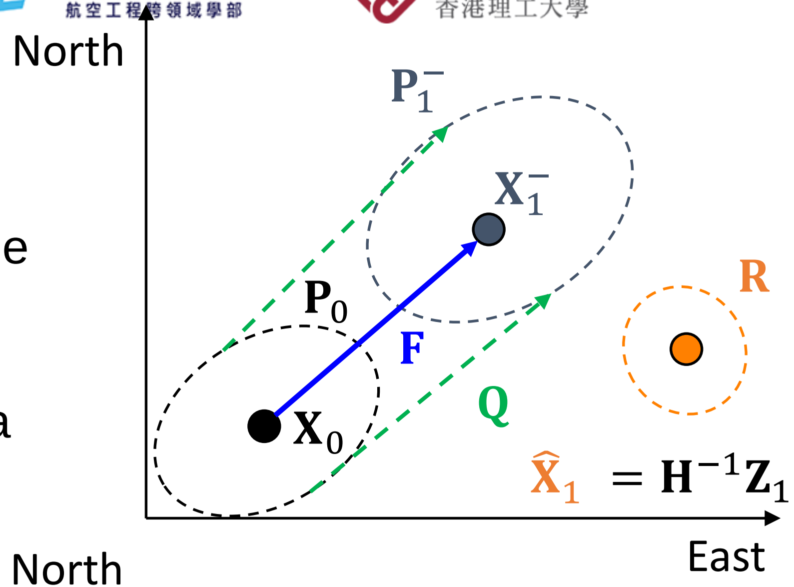


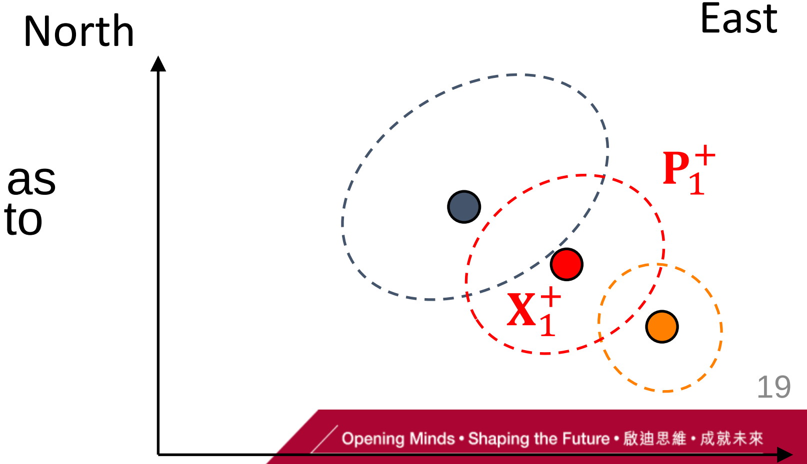
# Definitions

> **Z**, The *measurement vector* is a set of simultaneous measurements of properties of the system which are functions of the state vector.

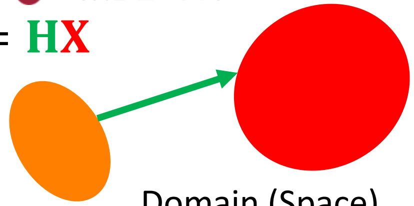
> **R**, Associated with the measurement vector is a *measurement noise covariance* matrix which describes the statistics of the noise on the measurements.

> **H, Z=H(X)=HX**, The *measurement model* describes how the measurement vector varies as a function of the true state vector (as opposed to the state vector estimate) in the absence of measurement noise.



# Key Equations of Kalman Filter




- $\mathbf{X}_t^- = \mathbf{F}\mathbf{X}_{t-1}^+ + \mathbf{B}\mathbf{U}_t$
- $\mathbf{P}_t^- = \mathbf{F}\mathbf{P}_{t-1}^+\mathbf{F}^\mathrm{T} + \mathbf{Q}$

**State Propagation**

- $\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H}\mathbf{X}_t^-$
- $\mathbf{S}_t = \mathbf{H}\mathbf{P}_{t-1}^-\mathbf{H}^\mathrm{T} + \mathbf{R}$
- $\mathbf{K}_t = \mathbf{P}_{t-1}^-\mathbf{H}^\mathrm{T}\mathbf{S}_t^{-1}$
- $\mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t\Delta\mathbf{Z}_t$
- $\mathbf{P}_t^+ = (\mathbf{I} - \mathbf{K}_t\mathbf{H})\mathbf{P}_{t-1}^-$

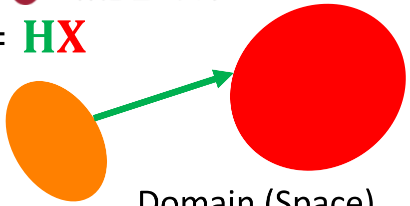
**Measurement Update**

# Measurement Innovation, $\Delta\mathbf{Z}_t$

- $\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H}\mathbf{X}_t^-$
- Meaning: The difference between propagation and measurement in the domain of $\mathbf{Z}$ (measurement)
- $\Delta\mathbf{Z}_t = 0$, What does it mean?
- $\Delta\mathbf{Z}_t \neq 0$, What shall we do?

# Kalman Filter Gain, $\mathbf{K}_t$

$\mathbf{Z} = \mathbf{H}\mathbf{X}$



Domain (Space) Transformation

$\mathbf{Z}^{\mathrm{T}} = \mathbf{X}^{\mathrm{T}}\mathbf{H}^{\mathrm{T}}$

> $\mathbf{K}_t = \mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}}\mathbf{S}_t^{-1}$

> $\mathbf{S}_t = \mathbf{H}\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}} + \mathbf{R}$

If we forget there is no denominator in Matrix…

> $\mathbf{K}_t = \dfrac{\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}}}{\mathbf{H}\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}}+\mathbf{R}}$

> $\mathbf{K}_t = \dfrac{(\mathbf{F}\mathbf{P}_{t-1}^{+}\mathbf{F}^{\mathrm{T}}+\mathbf{Q})\mathbf{H}^{\mathrm{T}}}{\mathbf{H}\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}}+\mathbf{R}}$

The propagated state covariance (uncertainty) in the domain of the transpose of **Z** (measurement)

The propagated state covariance (uncertainty) in the domain of **Z** (measurement)
\+ measurement covariance (uncertainty) **R**

The updated state covariance (uncertainty)
considering (+) propagation covariance (uncertainty)

# Kalman Filter Gain, $\mathbf{K}_t$

- $\mathbf{K}_t = \frac{(\mathbf{F}\mathbf{P}_{t-1}^{+}\mathbf{F}^{\mathrm{T}} + \mathbf{Q})\mathbf{H}^{\mathrm{T}}}{\mathbf{H}\mathbf{P}_{t-1}^{-}\mathbf{H}^{\mathrm{T}} + \mathbf{R}}$
- $\mathbf{R} \gg \mathbf{Q}$, what does it mean?
- $\mathbf{R} \ll \mathbf{Q}$, what does it mean?
- $\mathbf{R} = \mathbf{Q}$, what does it mean?

# Simplified the Models to Identity Matrix, $\mathbf{I}$

$\mathbf{F} = \mathbf{I}$
$\mathbf{H} = \mathbf{I}$
$\mathbf{B} = \mathbf{I}$

- $\mathbf{X}_t^- = \mathbf{F}\mathbf{X}_{t-1}^+ + \mathbf{B}\mathbf{U}_t$
- $\mathbf{P}_t^- = \mathbf{F}\mathbf{P}_{t-1}^+\mathbf{F}^{\mathrm{T}} + \mathbf{Q}$
- $\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H}\mathbf{X}_t^-$
- $\mathbf{S}_t = \mathbf{H}\mathbf{P}_{t-1}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R}$
- $\mathbf{K}_t = \mathbf{P}_{t-1}^-\mathbf{H}^{\mathrm{T}}\mathbf{S}_t^{-1}$
- $\mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t\Delta\mathbf{Z}_t$
- $\mathbf{P}_t^+ = (\mathbf{I} - \mathbf{K}_t\mathbf{H})\mathbf{P}_{t-1}^-$

→

- $\mathbf{X}_t^- = \mathbf{X}_{t-1}^+ + \mathbf{U}_t$
- $\mathbf{P}_t^- = \mathbf{P}_{t-1}^+ + \mathbf{Q}$
- $\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{X}_t^-$
- $\mathbf{S}_t = \mathbf{P}_{t-1}^- + \mathbf{R}$
- $\mathbf{K}_t = \mathbf{P}_{t-1}^-\mathbf{S}_t^{-1}$
- $\mathbf{X}_t^+ = \mathbf{X}_t^- + \mathbf{K}_t\Delta\mathbf{Z}_t$
- $\mathbf{P}_t^+ = (\mathbf{I} - \mathbf{K}_t)\mathbf{P}_{t-1}^-$

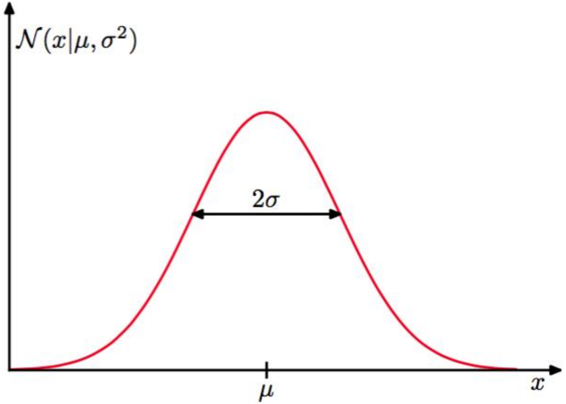
# How do we describe the uncertainty?

> Gaussian distribution

$$p(x) \sim N(\mu, \sigma^2)$$

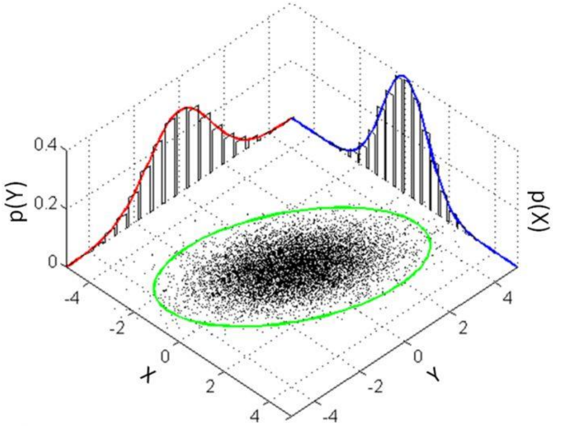
1D (univariate)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

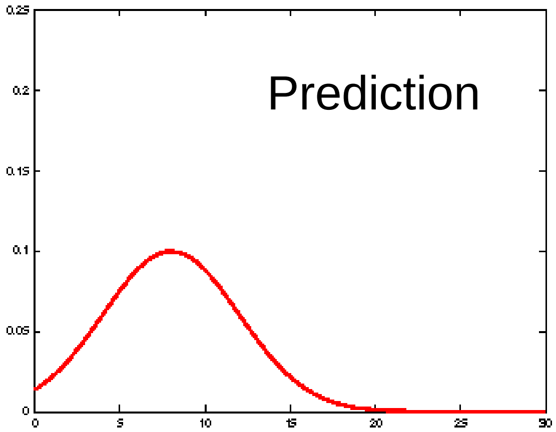


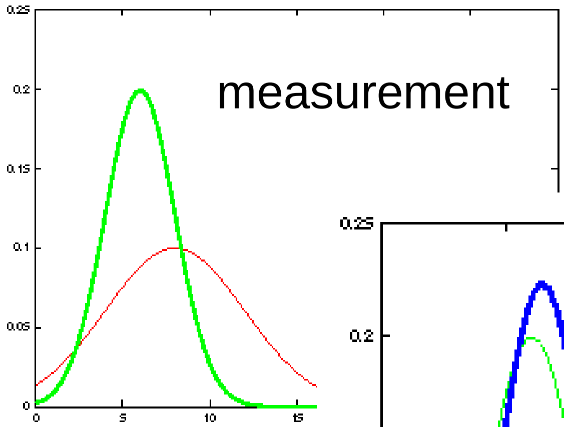

2D+ (multi variate)

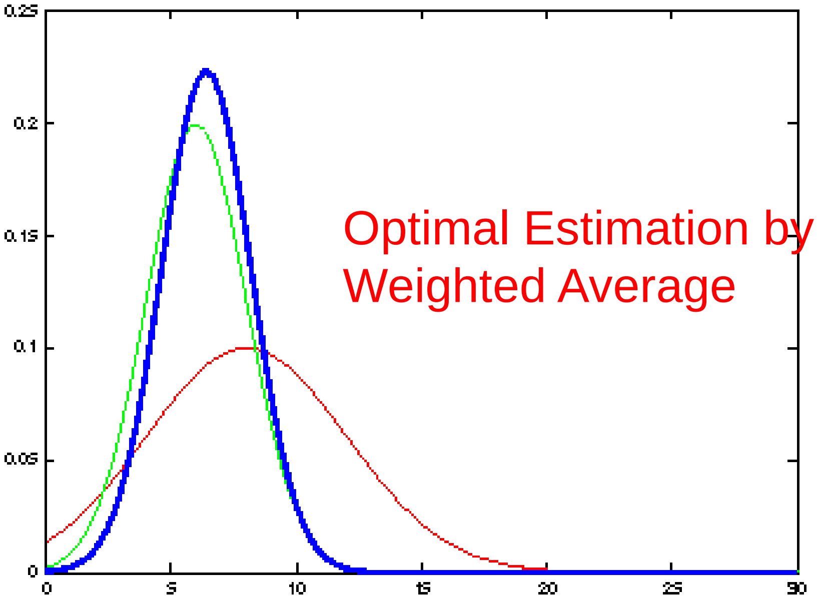
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\mathbf{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$



# What Kalman Filter tries to achieve?



# Fusion of Gaussian distribution in 1D

> Distribution of two measurements

$$\hat{q}_1 = q_1 \text{ with variance } \sigma_1^2$$
$$\hat{q}_2 = q_2 \text{ with variance } \sigma_2^2$$

> Weighted least-square

$$S = \sum_{i=1}^{n} w_i(\hat{q} - q_i)^2$$

> Finding minimum error

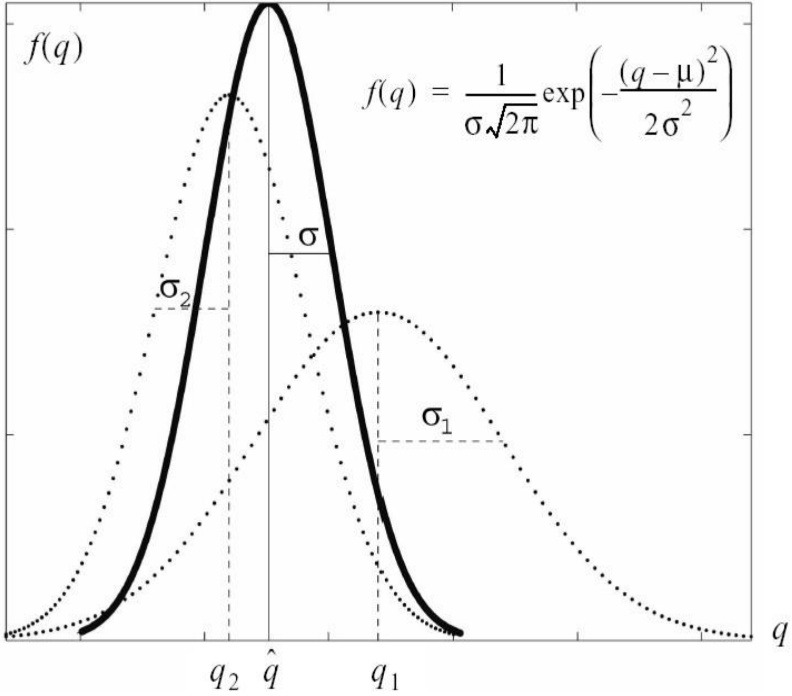
If $w_i = 1/\sigma_i^2$

$$\frac{\partial S}{\partial \hat{q}} = \frac{\partial}{\partial \hat{q}} \sum_{i=1}^{n} w_i(\hat{q} - q_i)^2 = 2\sum_{i=1}^{n} w_i(\hat{q} - q_i) = 0$$

After some calculation and rearrangements

$$\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(q_2 - q_1)$$

Kalman Gain



# Question: Why Kalman Filter is Optimal?

> What does it mean to Optimal?
- Maximum the Probability of the Estimation!

> What assumptions are made in Kalman Filter so that it can achieve Optimal?

A. Gaussian Random Variable (Gaussian Noise)

B. $1^{st}$ Order of Markov Chain

# Kalman filter in GNSS

- **Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter**
- Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter

# Statistical Methods for Estimation and Optimization

**Statistical method**

**States set**

$$\chi = \{\mathbf{x}_o, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k\}$$

$$\mathbf{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_m \end{bmatrix}$$

**Measurement set**

$$\mathbf{Z} = \{\mathbf{z}_o, \mathbf{z}_1, \mathbf{z}_2, \cdots, \mathbf{z}_k\}$$

$$\mathbf{z} = \begin{bmatrix} z_0 \\ \vdots \\ z_n \end{bmatrix}$$

**Frequentist** → **Maximum Likelihood Estimation (MLE)**

$$P(\mathbf{z}|\mathbf{x})$$

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}\in\mathbf{X}} P_{(\mathbf{Z}|\chi)}(\mathbf{z}|\mathbf{x})$$

event-centric estimation, fully based on data. Should be used in Big Data application.

**Bayesians** → **Maximum a Posterior Estimation (MAP)**

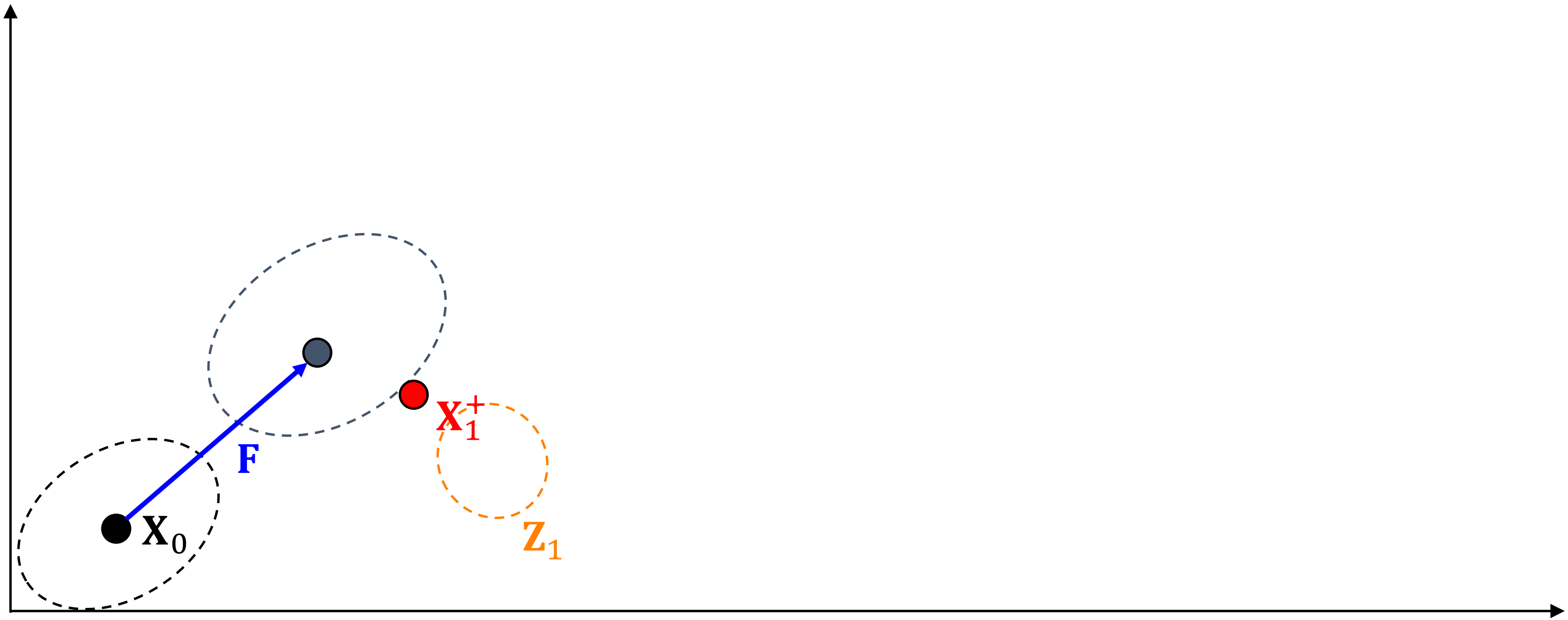
$$P(\mathbf{x}|\mathbf{z}) = \frac{P(\mathbf{z}|\mathbf{x})P(\mathbf{x})}{P(\mathbf{z})}$$

$$\hat{\mathbf{x}} = \arg\max_{\mathbf{x}\in\mathbf{X}} P_{(\mathbf{Z}|\chi)}(\mathbf{x}|\mathbf{z})$$

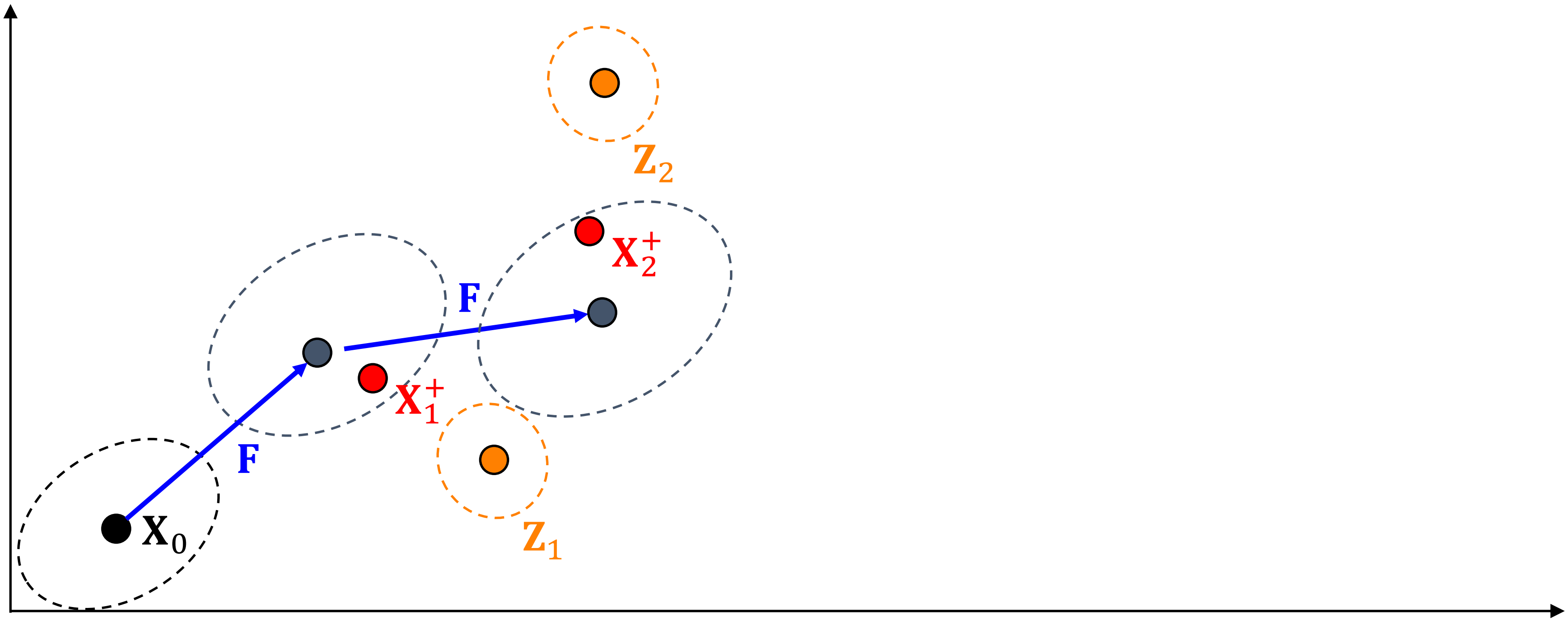
observer-centric estimation, required a knowledgeful observer (good prior-information)

**Optimization, i.e., Kalman filter, factor graph optimization, etc.**

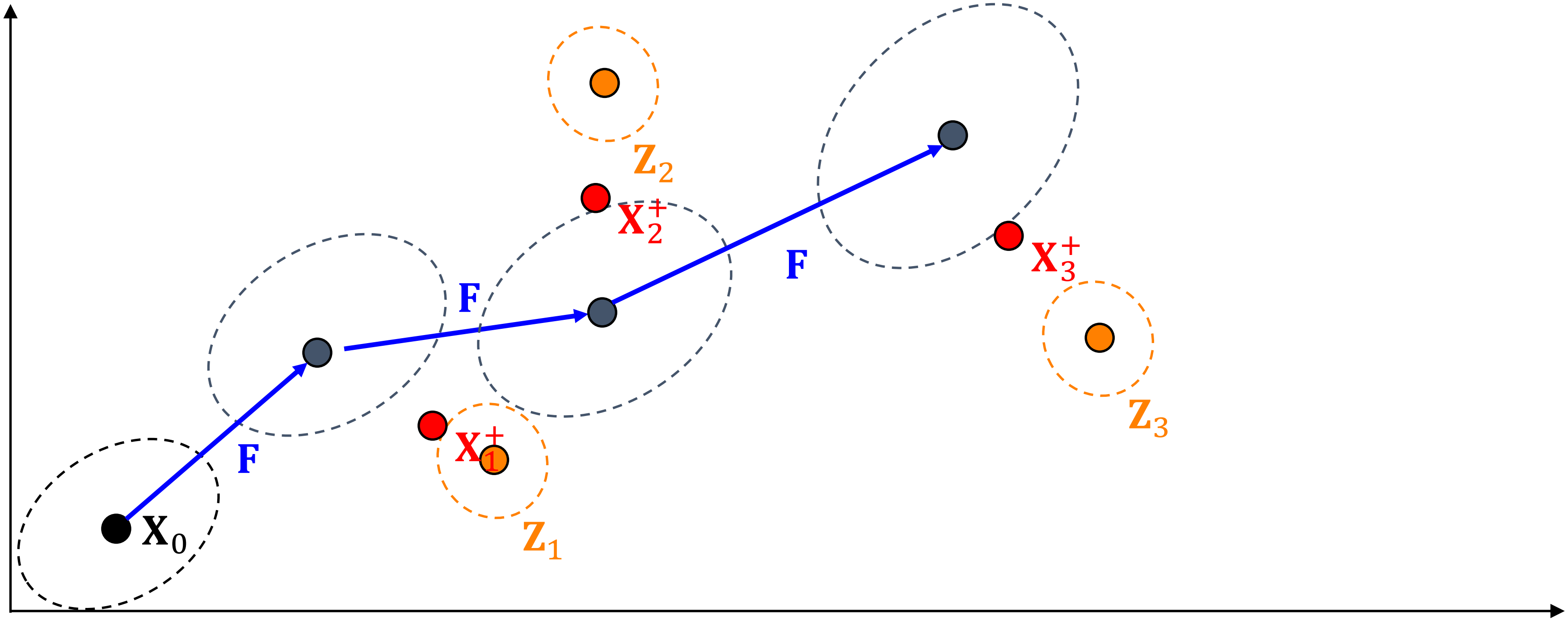
# Batch (including all the data in the past) optimization



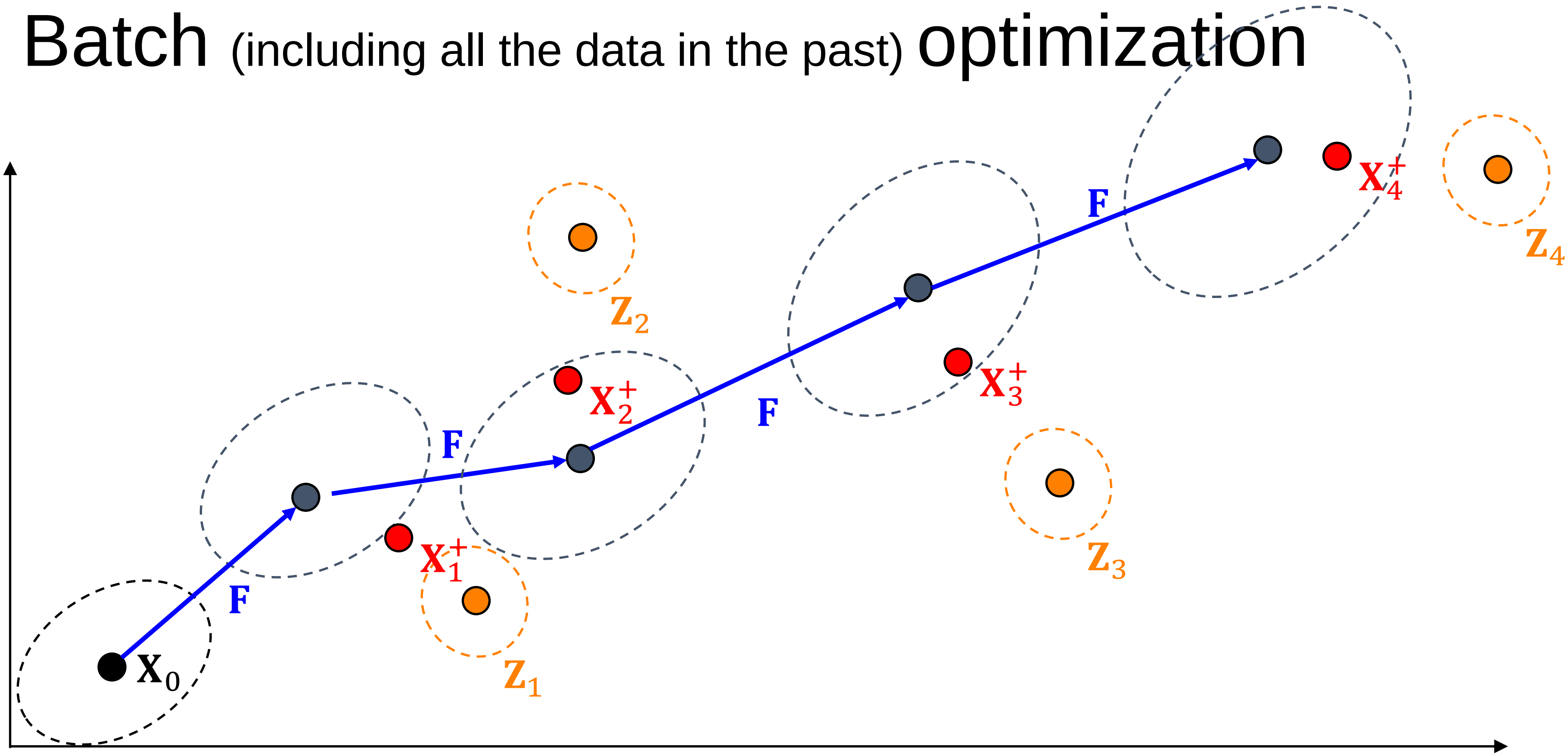
# Batch (including all the data in the past) optimization



# Batch (including all the data in the past) optimization

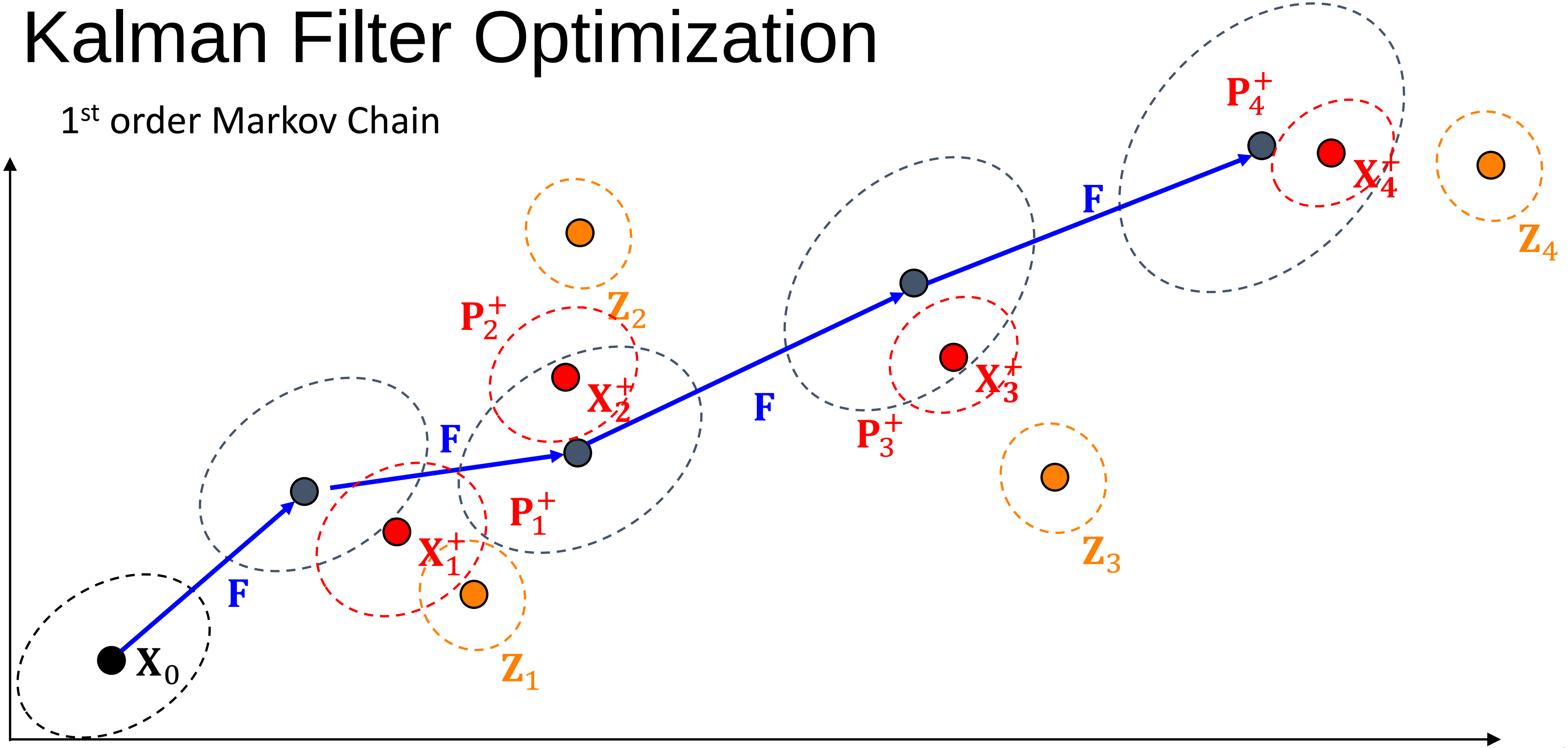


# Batch (including all the data in the past) optimization

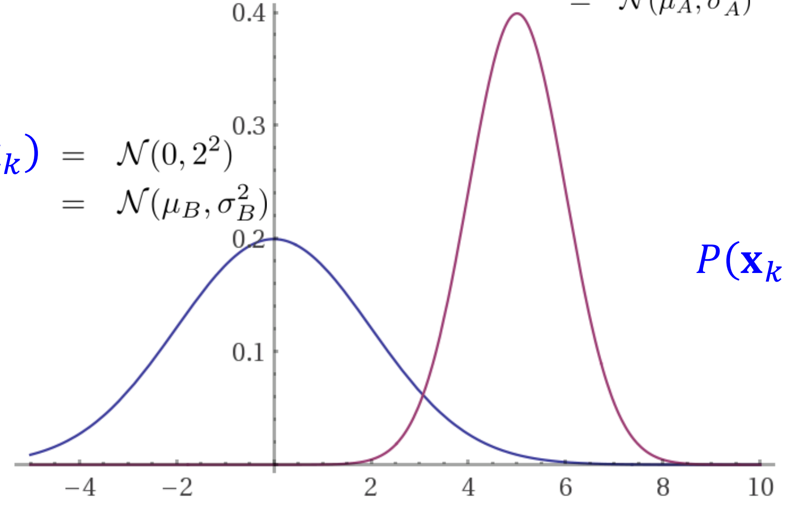


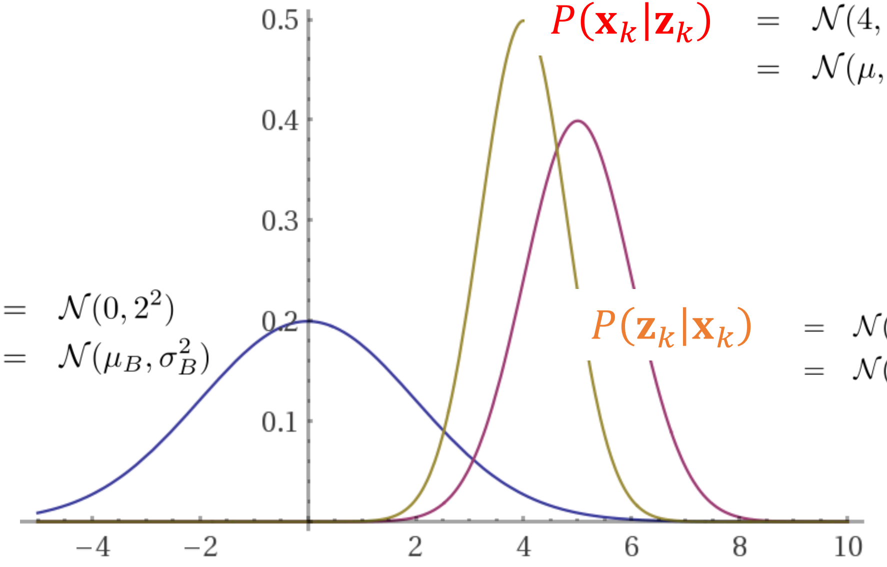
# Kalman Filter Optimization

1st order Markov Chain



# Kalman Filter with 1D state——Update step



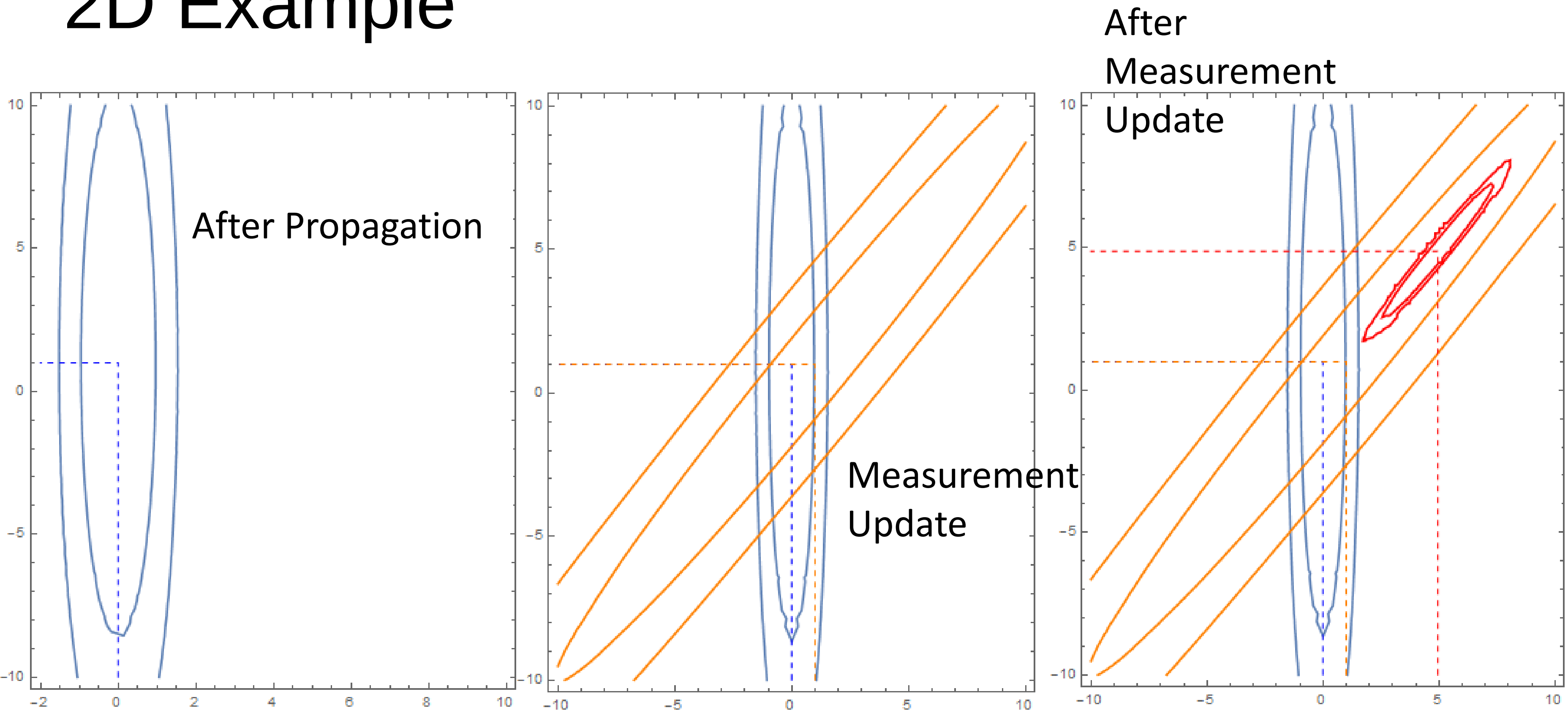

Kalman Gain: specifies how much effect will the measurement have in the posterior, compared to the prediction prior.

Which one do you trust more, your prior or your measurement ?

$$\mu = \mu_B + \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}(\mu_A - \mu_B)$$

$$\sigma^2 = \sigma_B^2 - \frac{\sigma_B^2}{\sigma_A^2 + \sigma_B^2}\sigma_B^2$$

# 2D Example


After Propagation
Measurement Update
After Measurement Update

# Kalman filter in GNSS

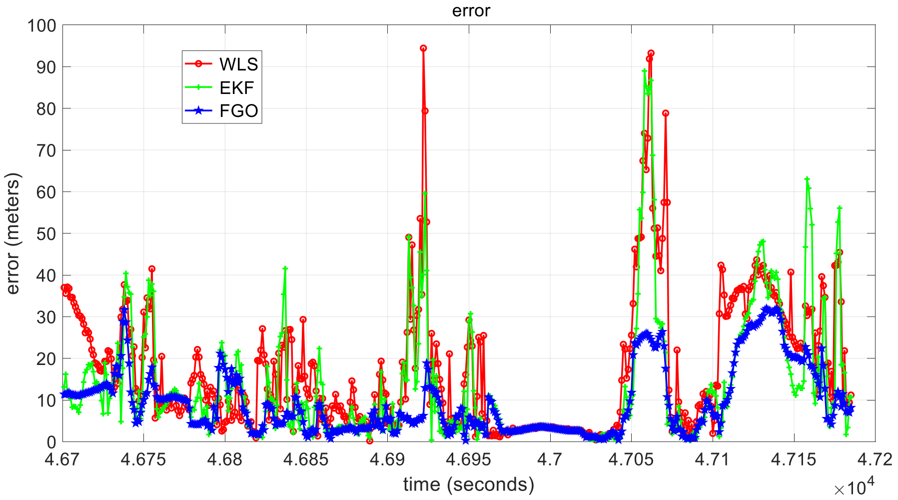
- Example of the GNSS loosely-coupled pseudorange/Doppler integration using Kalman filter
- **Example of the GNSS tightly-coupled pseudorange/Doppler integration using Kalman filter**

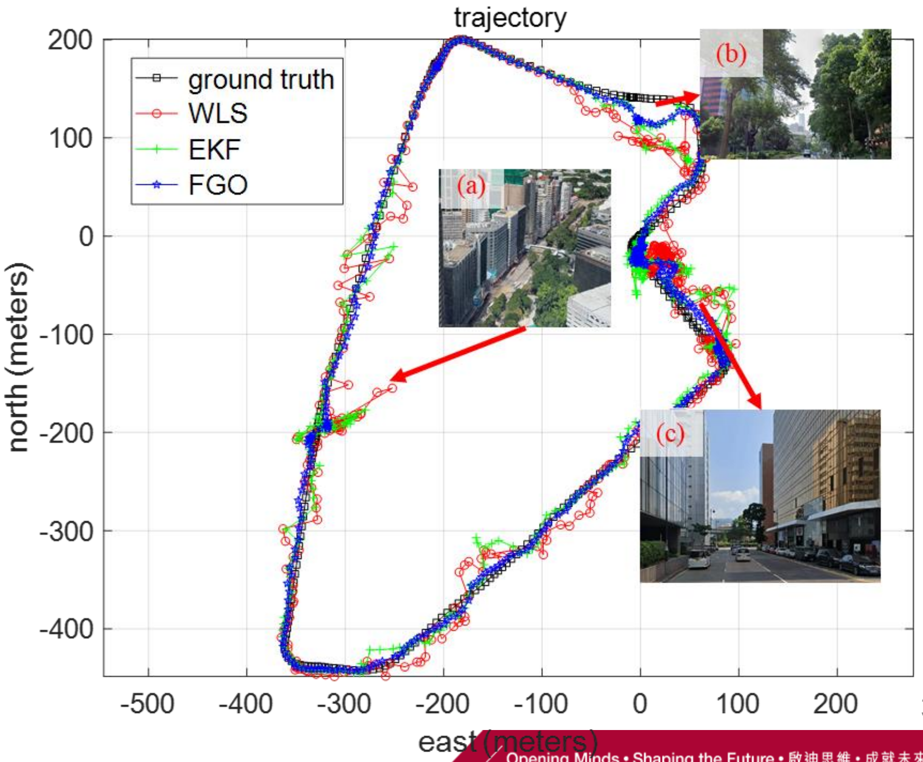
# Evaluation of GNSS Positioning

WLS*: weighted least square with pseudorange
EKF*: Pseudorange/Doppler fusion with extended Kalman filter
FGO*: Pseudorange/Doppler fusion with factor graph optimization

GNSS positioning performance using the three listed methods

| All data | WLS | EKF | FGO |
|---|---|---|---|
| **MEAN (m)** | 17.39 | 13.61 | 9.45 |
| **STD (m)** | 16.01 | 15.19 | 8.06 |
| **MAX (m)** | 94.43 | 88.97 | 31.94 |
| **Availability** | 100% | 100% | 100% |



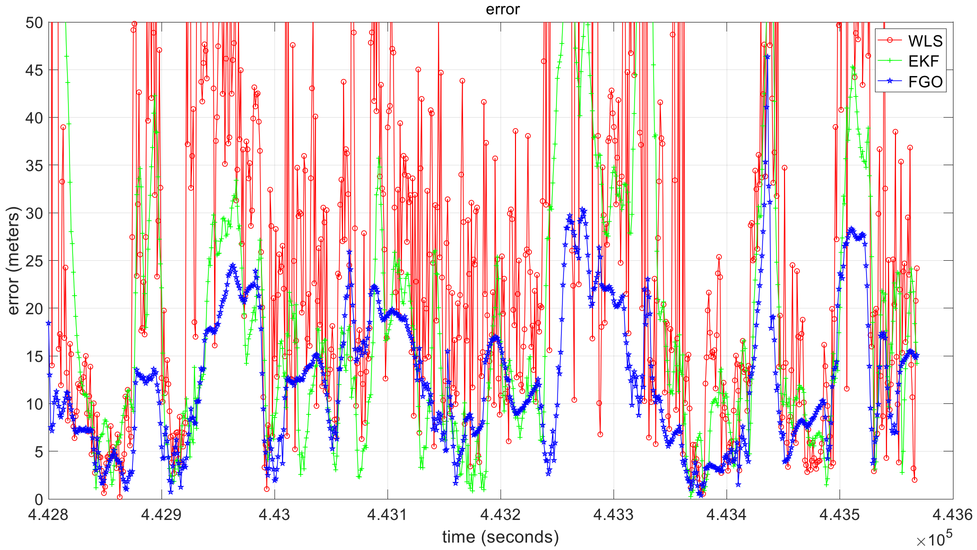



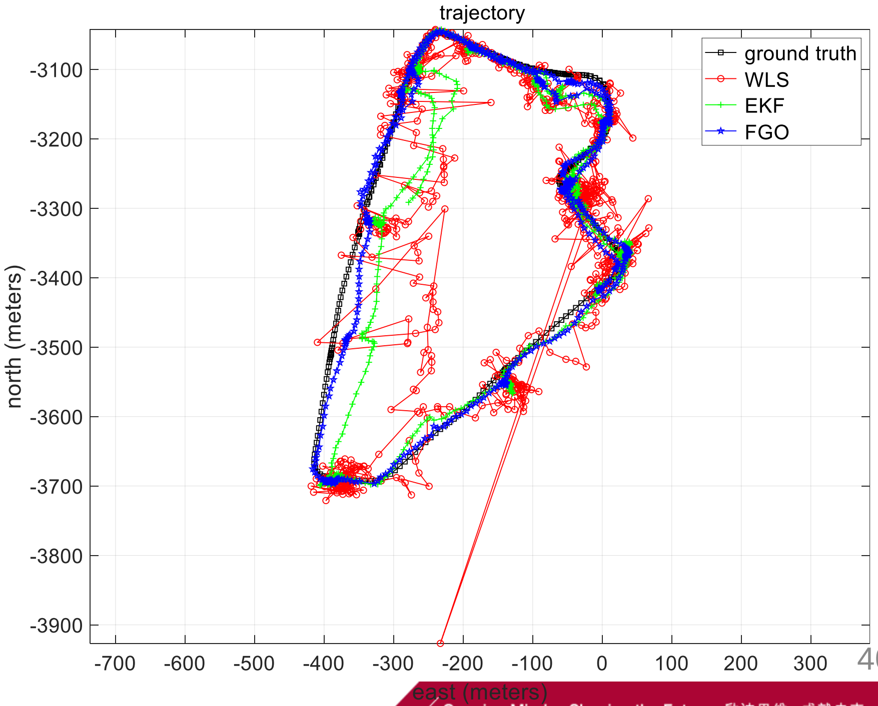
# Evaluation with Huawei P40 Pro

Huawei P40 Pro Phone

WLS*: weighted least square with pseudorange
EKF*: Pseudorange/Doppler fusion with extended Kalman filter
FGO*: Pseudorange/Doppler fusion with factor graph optimization

| All data | WLS | EKF | FGO |
|---|---|---|---|
| **MEAN (m)** | 31.98 | 19.84 | 12.541 |
| **STD (m)** | 38.22 | 15.78 | 7.48 |
| **MAX (m)** | 701.7 | 77.28 | 46.36 |



# Q&A

## Thank you for your attention ☺

## Q&A

Dr. Weisong Wen

If you have any questions or inquiries, please feel free to contact me.

Email: welson.wen@polyu.edu.hk

# Supplementary: GNSS/INS Integration Using Kalman Filtering

# Inertial navigation system

$\hat{\mathbf{a}}_t$ , $\hat{\boldsymbol{\omega}}_t$ are the raw accelerometer and gyroscope measurements in the body frame

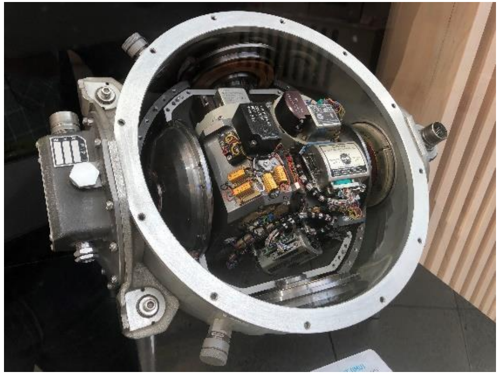
$\mathbf{a}_t$ , $\boldsymbol{\omega}_t$ are expected measurements

The cap ^ denotes the noisy measurement or estimation of a certain quantity

$$\hat{\mathbf{a}}_t = \mathbf{a}_t + \mathbf{R}_w^t \mathbf{g}^w + \mathbf{b}_{a_t} + \mathbf{n}_a \quad (1)$$

$$\hat{\boldsymbol{\omega}}_t = \boldsymbol{\omega}_t + \mathbf{b}_{\omega_t} + \mathbf{n}_\omega \quad (2)$$

$$\mathbf{n}_a \sim \mathcal{N}(0, \sigma_a^2), \mathbf{n}_\omega \sim \mathcal{N}(0, \sigma_\omega^2)$$

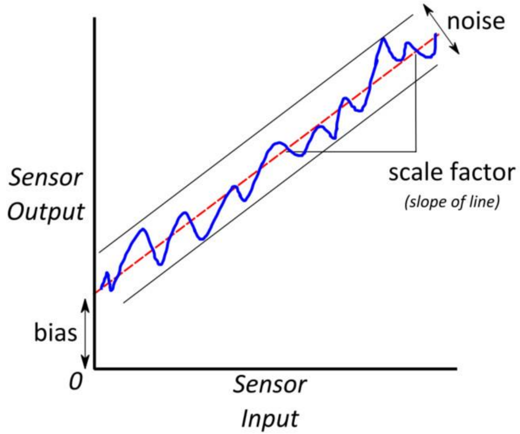


# Error analysis of inertial navigation system

> The errors of **accelerometer and gyroscope** can be divided into: **deterministic error & random error.**

> Deterministic errors can be calibrated in advance including bias, scale...

> Random error usually assumes that noise obeys Gaussian distribution, including Gaussian white noise, bias random walk...



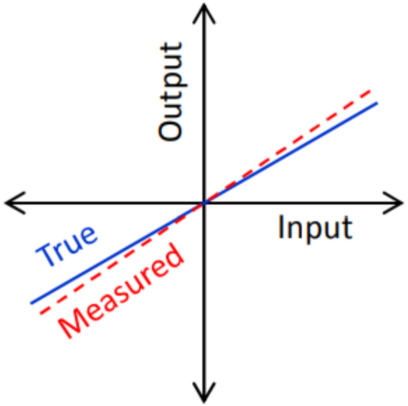

**Common systematic errors in IMU**
bias
noise
scale factor

# Deterministic error
(sourcing imperfectness of electrical/mechanical components)

- Bias: In theory, the output of the IMU sensor should be 0 when there is no external action. However, there is a bias $\mathbf{b}$ to the international data. Influence of accelerometer bias on orientation estimation:

$$\mathbf{V}_{\mathbf{error}} = \mathbf{b}_{\mathbf{a}} t, \mathbf{P}_{\mathbf{error}} = \frac{1}{2} \mathbf{b}_{\mathbf{a}} t^2$$

- Scale: The ratio between the actual value and the sensor output value.



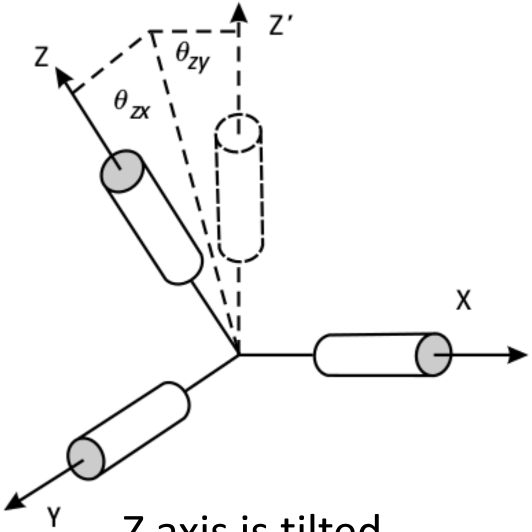
# Deterministic error (sourcing imperfectness of installation)

> Nonorthogonality/Misalignment Errors: When manufacturing multi-axis IMU sensors, due to the manufacturing process, the xyz axis may not be vertical.

Scale + Misalignment

$$\begin{bmatrix} l_{ax} \\ l_{ay} \\ l_{az} \end{bmatrix} = \begin{bmatrix} s_{xx} & m_{xy} & m_{xz} \\ m_{yz} & s_{yy} & m_{yz} \\ m_{zx} & m_{zy} & s_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

Measured Acc                                True Acc




Z axis is tilted

# Deterministic error calibration method—Accelerometer

> The six-sided method means that the three axes of the accelerometer are placed horizontally up or down for a period of time, and data on the six sides are collected to complete the calibration.

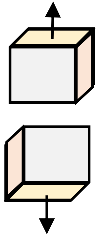
If the axes are orthogonal, it is easy to get bias and scale:

$$\mathbf{b_a} = \frac{\mathbf{l_f^{up}} + \mathbf{l_f^{down}}}{2}$$

$$\mathbf{s_a} = \frac{\mathbf{l_f^{up}} - \mathbf{l_f^{down}}}{2 \cdot \mathbf{g}} = \begin{bmatrix} s_{a,xx} \\ s_{a,yy} \\ s_{a,zz} \end{bmatrix}$$

$\mathbf{l_f^{up}}$

$\mathbf{l_f^{dowb}}$



$\mathbf{l}$ is the measured value of a certain axis of the accelerometer, $\mathbf{g}$ is the local gravity acceleration

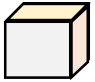
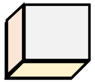
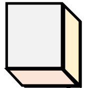
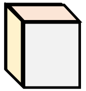
# Deterministic error calibration method—Accelerometer

- When considering the inter-axis error, the relationship between the actual acceleration and the measured value is:

$$\begin{bmatrix} l_{ax} \\ l_{ay} \\ l_{az} \end{bmatrix} = \begin{bmatrix} s_{xx} & m_{xy} & m_{xz} \\ m_{yz} & s_{yy} & m_{yz} \\ m_{zx} & m_{zy} & s_{zz} \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} + \begin{bmatrix} b_{ax} \\ b_{ay} \\ b_{az} \end{bmatrix}$$

- When placed horizontally and statically on 6 sides, the theoretical value of acceleration is

$$\mathbf{a_1} = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_2} = \begin{bmatrix} -g \\ 0 \\ 0 \end{bmatrix}, \mathbf{a_3} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \mathbf{a_4} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix}, \mathbf{a_5} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}, \mathbf{a_6} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

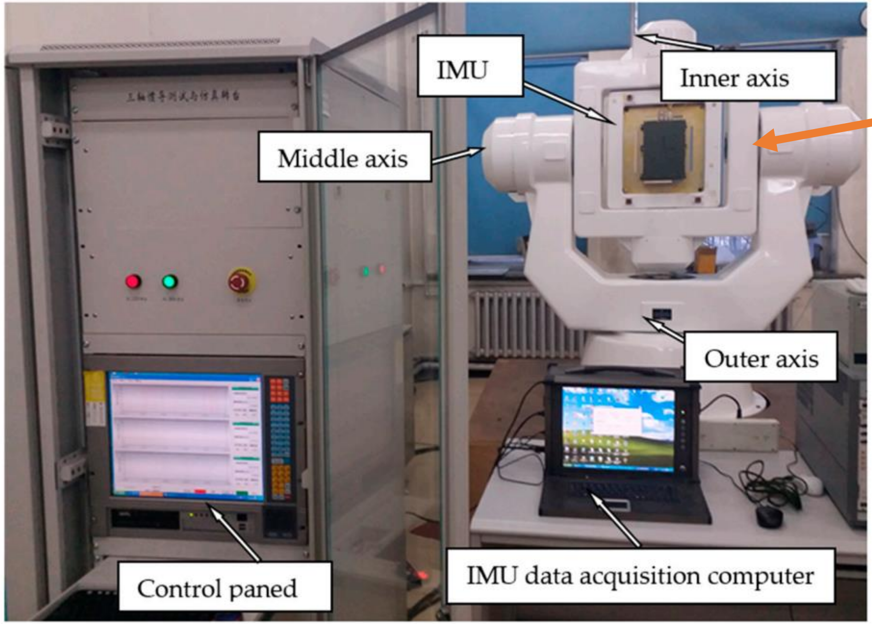
- Corresponding measurement value matrix $\mathbf{L}$

$$\mathbf{L} = [\boldsymbol{l}_1 \quad \boldsymbol{l}_2 \quad \boldsymbol{l}_3 \quad \boldsymbol{l}_4 \quad \boldsymbol{l}_5 \quad \boldsymbol{l}_6]$$

- 12 variables can be obtained by using least squares.

# Deterministic error calibration method——Gyroscope

> Unlike the six-sided method of accelerometer, the true value of the gyroscope is provided by a high-precision turntable. The 6 faces in this refer to the clockwise and counterclockwise rotation of each axis

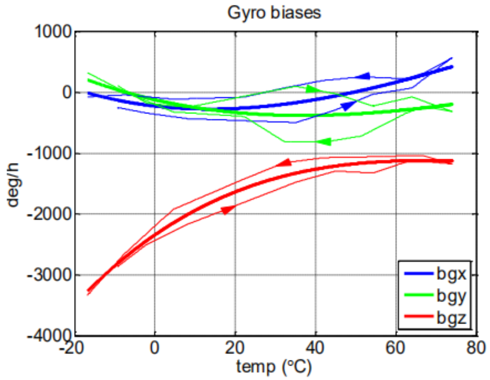


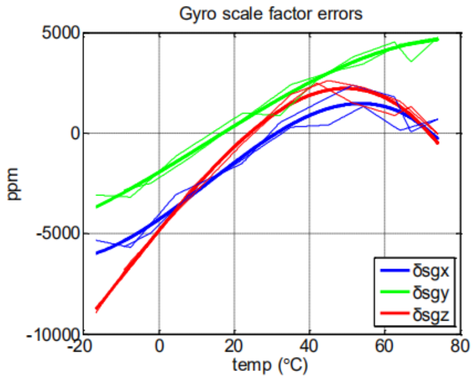

high-precision three-axis turntable

# Random error – Unstableness of electrical and mechanical component due to temperature

- We can calibrate to do temperature compensation on the bias and scale estimated by the sensor, and to obtain the values of bias and scale at different temperatures and draw them into a curve.
- Soak method: control the temperature value of **the constant temperature room**, and then read the sensor value for calibration.

The thin solid lines are the results under separated heating and cooling processes The thick lines are the final curve fitted result .

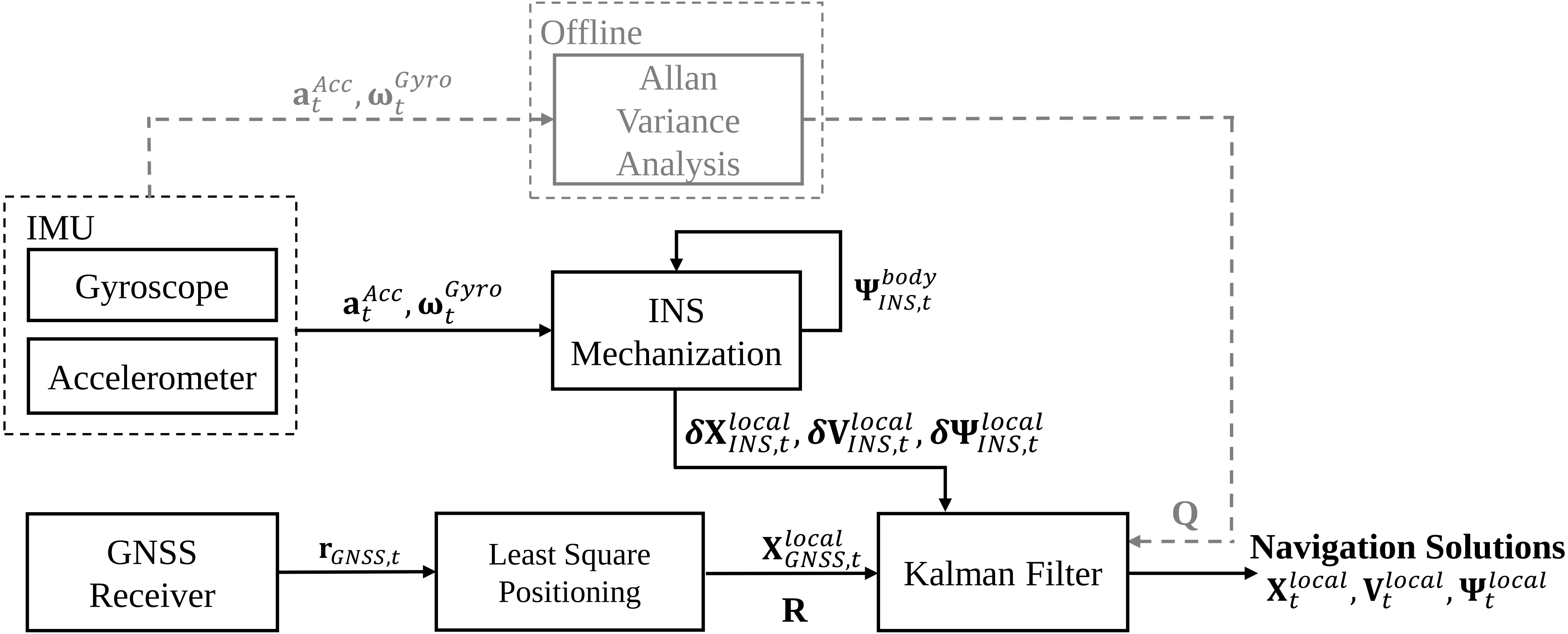


# Nomenclature

- $\mathbf{a}_t^{Acc}$ :measured 3-axis accelerations by the accelerometers at epoch $t$
- $\boldsymbol{\omega}_t^{Gyro}$:measured 3-axis rotation by the gyroscopes at epoch $t$
- $\mathbf{X}_{INS,t}^{body}$ :estimated 3-axis position in body frame by INS at epoch $t$
- $\mathbf{V}_{INS,t}^{body}$:estimated 3-axis velocity in body frame by INS at epoch $t$
- $\boldsymbol{\Psi}_{INS,t}^{body}$:estimated 3-axis orientation in body frame (Euler angles) by INS at epoch $t$
- $\mathbf{B}_{a,t}^{body}$ :estimated 3−axis biaes of accelerometers in body frame at epoch $t$
- $\mathbf{B}_{\omega,t}^{body}$ :estimated 3−axis biaes of gyroscopes in body frame at epoch $t$
- $W_{b_a}$:estimated 3−axis random walk noise of accelerometers in body frame
- $W_{b_a}$:estimated 3−axis random walk noise of gyroscopes in body frame

# Open Loop


Offline
Allan Variance Analysis
$\mathbf{a}_t^{Acc}, \boldsymbol{\omega}_t^{Gyro}$
IMU
Gyroscope
Accelerometer
$\mathbf{a}_t^{Acc}, \boldsymbol{\omega}_t^{Gyro}$
INS Mechanization
$\boldsymbol{\Psi}_{INS,t}^{body}$
$\boldsymbol{\delta}\mathbf{X}_{INS,t}^{local}, \boldsymbol{\delta}\mathbf{V}_{INS,t}^{local}, \boldsymbol{\delta}\boldsymbol{\Psi}_{INS,t}^{local}$
Q
GNSS Receiver
$\mathbf{r}_{GNSS,t}$
Least Square Positioning
$\mathbf{X}_{GNSS,t}^{local}$
R
Kalman Filter
Navigation Solutions
$\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \boldsymbol{\Psi}_t^{local}$

# INS Mechanization

Sola, Joan. "Quaternion kinematics for the error-state Kalman filter." *arXiv preprint arXiv:1711.02508* (2017).

The Euler angle rates obtained by angular velocity:

$$\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \dot{\boldsymbol{\Psi}}_{INS,t}^{body} = \begin{bmatrix}1 & sin(\phi)\,tan(\theta) & cos(\phi)\,tan(\theta)\\ 0 & cos(\phi) & -sin(\phi)\\ 0 & sin(\phi)\,sec(\theta) & cos(\phi)\,sec(\theta)\end{bmatrix}\boldsymbol{\omega}_t^{Gyro} = \begin{bmatrix}p\\ q\\ r\end{bmatrix}$$

$$\boldsymbol{\delta\Psi}_{INS,t}^{body} = \dot{\boldsymbol{\Psi}}_{INS,t}^{body}\Delta t = \begin{bmatrix}\delta\phi\\ \delta\theta\\ \delta\psi\end{bmatrix}$$

Rotate the Euler angles from Body to Local

$$\boldsymbol{\Psi}_{INS,t}^{local} = \mathbf{R}_{body}^{local}\boldsymbol{\Psi}_{INS,t}^{body}$$

Update the current Euler angles

$$\boldsymbol{\Psi}_{INS,t}^{body} = \boldsymbol{\Psi}_{INS,t-1}^{body} + \boldsymbol{\delta\Psi}_{INS,t}^{body}$$

$$\mathbf{R}_{body}^{local} = \mathbf{R}(X,\phi)\mathbf{R}(Y,\theta)\mathbf{R}(Z,\psi) = \begin{bmatrix}1 & 0 & 0\\ 0 & cos(\phi) & -sin(\phi)\\ 0 & sin(\phi) & cos(\phi)\end{bmatrix}\begin{bmatrix}cos(\theta) & 0 & sin(\theta)\\ 0 & 1 & 0\\ -sin(\theta) & 0 & cos(\theta)\end{bmatrix}\begin{bmatrix}cos(\psi) & -sin(\psi) & 0\\ sin(\psi) & cos(\psi) & 0\\ 0 & 0 & 1\end{bmatrix}$$

# INS Mechanization

To remove the gravity from the acceleration

$$\mathbf{a}_{INS,t}^{local} = \mathbf{R}_{body}^{local}\mathbf{a}_{INS,t}^{body} - \mathbf{g}$$

To obtain the change of aircraft in terms of position, velocity and orientation

$$\boldsymbol{\delta}\mathbf{X}_{INS,t}^{local} = \mathbf{X}_{INS,t}^{local} - \mathbf{X}_{INS,t-1}^{local} = \mathbf{V}_{INS,t-1}^{local}\Delta t + \frac{\boldsymbol{a}_{INS,t}^{local}\Delta t^2}{2}$$

$$\boldsymbol{\delta}\mathbf{V}_{INS,t}^{local} = \mathbf{V}_{INS,t}^{local} - \mathbf{V}_{INS,t-1}^{local} = \boldsymbol{a}_{INS,t}^{local}\Delta t$$

$$\boldsymbol{\delta}\boldsymbol{\Psi}_{INS,t}^{local} = \boldsymbol{\Psi}_{INS,t}^{local} - \boldsymbol{\Psi}_{INS,t-1}^{local}$$

# Kalman Filter——GNSS/INS(Open Loop)

System States:

$$\mathbf{X}_t = (\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \mathbf{\Psi}_t^{local})$$

$$\mathbf{X}_t^{local} = (x_t^{local}, y_t^{local}, z_t^{local})$$

$$\mathbf{V}_t^{local} = (vx_t^{local}, vy_t^{local}, vz_t^{local})$$

$$\mathbf{\Psi}_t^{local} = (\phi_{roll}, \theta_{pitch}, \psi_{yaw})$$

Propagation model:

$$\mathbf{X}_t^- = \mathbf{F}\mathbf{X}_{t-1}^+ + \mathbf{B}\mathbf{U}_t$$

$$\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \mathbf{U}_t = \begin{bmatrix} \delta\mathbf{X}_{INS,t}^{local} \\ \delta\mathbf{V}_{INS,t}^{local} \\ \delta\mathbf{\Psi}_{INS,t}^{local} \end{bmatrix}$$

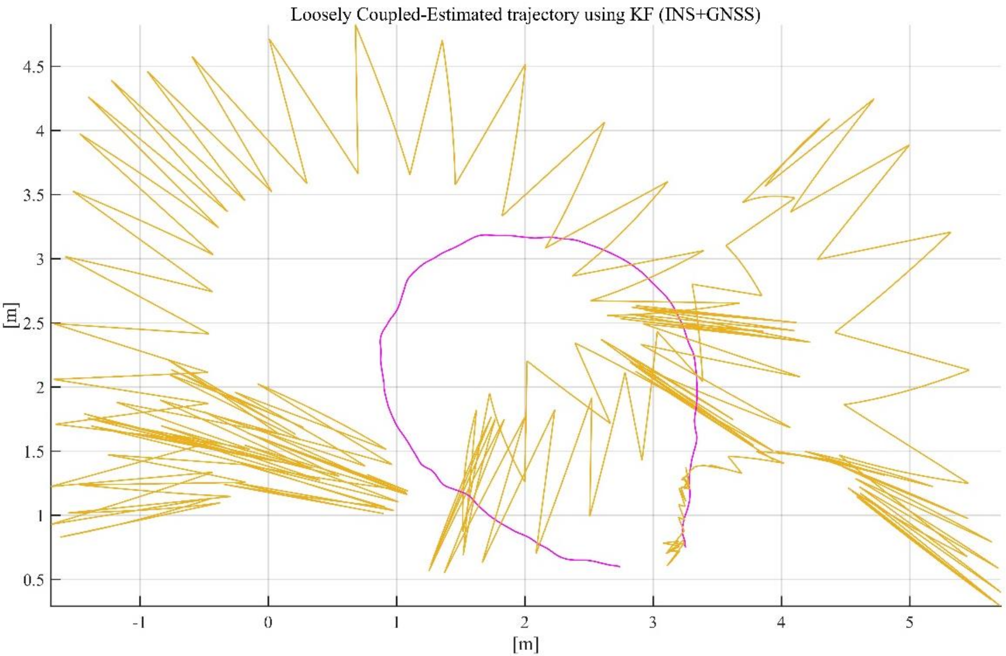
Measurement model:

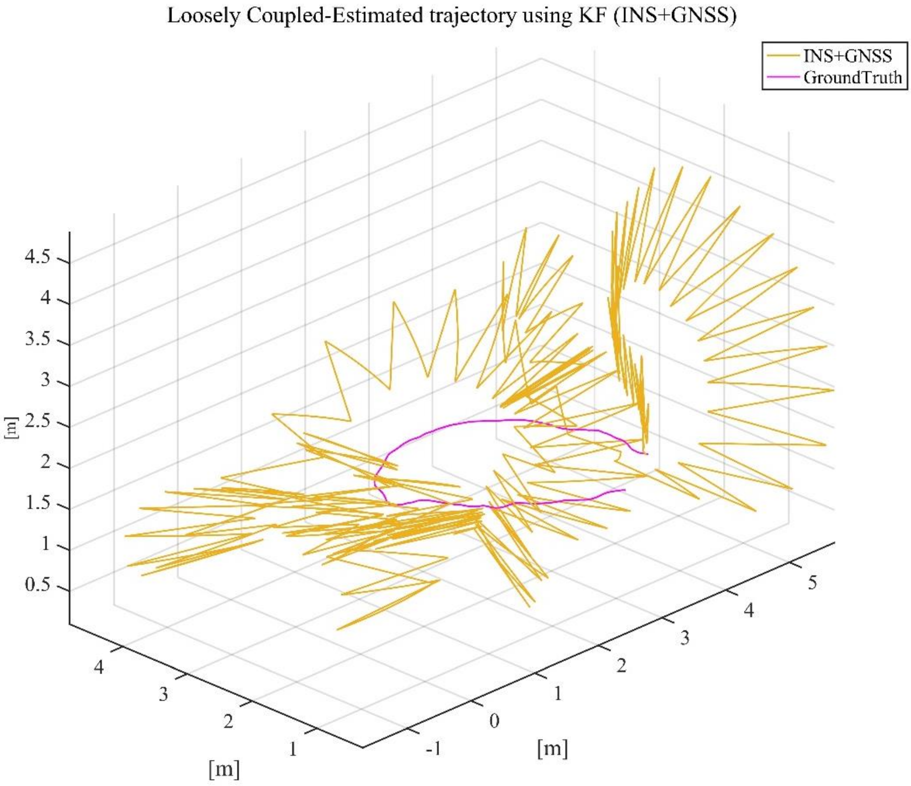
$$\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H}\mathbf{X}_t^-$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3\times3} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

$$\mathbf{Z}_t = \mathbf{X}_{GNSS,t}^{local} = \begin{bmatrix} x_{GNSS,t}^{local} \\ y_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \end{bmatrix}$$

# Open Loop

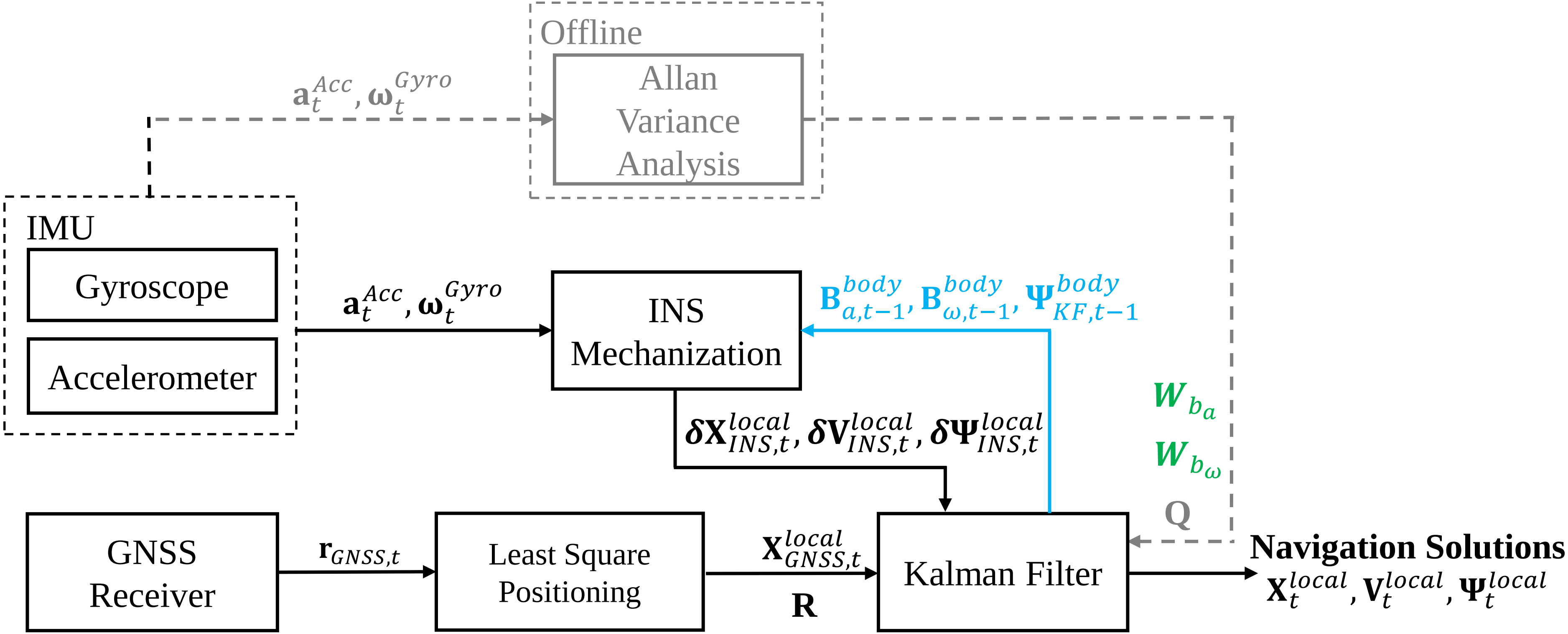

Loosely Coupled-Estimated trajectory using KF (INS+GNSS)
[m]
[m]


Loosely Coupled-Estimated trajectory using KF (INS+GNSS)
INS+GNSS
GroundTruth
[m]
[m]
[m]

# Closed-loop Correction

- The estimated position, velocity, and attitude errors are fed back to the inertial navigation processor, where they are used to correct the inertial navigation solution itself.
- any accelerometer and gyro errors estimated by the Kalman filter are fed back to correct the IMU measurements, as they are input to the inertial navigation equations.
- Unlike the position, velocity, and attitude corrections, the accelerometer and gyro corrections must be applied on every iteration

# Closed Loop


Offline
Allan Variance Analysis
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IMU
Gyroscope
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INS Mechanization
$\mathbf{B}_{a,t-1}^{body}, \mathbf{B}_{\omega,t-1}^{body}, \boldsymbol{\Psi}_{KF,t-1}^{body}$
$\boldsymbol{\delta}\mathbf{X}_{INS,t}^{local}, \boldsymbol{\delta}\mathbf{V}_{INS,t}^{local}, \boldsymbol{\delta\Psi}_{INS,t}^{local}$
$W_{ba}$
$W_{b\omega}$
$\mathbf{Q}$
GNSS Receiver
$\mathbf{r}_{GNSS,t}$
Least Square Positioning
$\mathbf{X}_{GNSS,t}^{local}$
$\mathbf{R}$
Kalman Filter
Navigation Solutions
$\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \boldsymbol{\Psi}_t^{local}$

# INS Mechanization

The Euler angle rates obtained by angular velocity:

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \dot{\boldsymbol{\Psi}}_{INS,t}^{body} = \begin{bmatrix} 1 & sin(\phi)\,tan(\theta) & cos(\phi)\,tan(\theta) \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi)\,sec(\theta) & cos(\phi)\,sec(\theta) \end{bmatrix} \left(\boldsymbol{\omega}_t^{Gyro} - \mathbf{B}_{\omega,t-1}^{body}\right)$$

$$\boldsymbol{\delta\Psi}_{INS,t}^{body} = \dot{\boldsymbol{\Psi}}_{INS,t}^{body} \Delta t = \begin{bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{bmatrix}$$

Update the current Euler angles

$$\boldsymbol{\Psi}_{INS,t}^{body} = \boldsymbol{\Psi}_{KF,t-1}^{body} + \boldsymbol{\delta\Psi}_{INS,t}^{body}$$

$$\boldsymbol{\Psi}_{KF,t-1}^{local} = \mathbf{R}_{body}^{local} \boldsymbol{\Psi}_{KF,t-1}^{body}$$

Rotate the Euler angles from Body to Local

$$\boldsymbol{\Psi}_{INS,t}^{local} = \mathbf{R}_{body}^{local} \boldsymbol{\Psi}_{INS,t}^{body}$$

$$\mathbf{R}_{body}^{local} = \mathbf{R}(X,\phi)\mathbf{R}(Y,\theta)\mathbf{R}(Z,\psi)$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) \\ 0 & sin(\phi) & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix} \begin{bmatrix} cos(\psi) & -sin(\psi) & 0 \\ sin(\psi) & cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# INS Mechanization

To remove the gravity from the acceleration

$$\mathbf{a}_{INS,t}^{local} = \mathbf{R}_{body}^{local}\left(\mathbf{a}_{INS,t}^{body} - \mathbf{B}_{a,t-1}^{body}\right) - \mathbf{g}$$

To obtain the change of aircraft in terms of position, velocity and orientation

$$\boldsymbol{\delta}\mathbf{X}_{INS,t}^{body} = \mathbf{V}_{INS,t-1}^{local}\Delta t + \frac{\boldsymbol{a}_{INS,t}^{local}\Delta t^2}{2}$$

$$\boldsymbol{\delta}\mathbf{V}_{INS,t}^{local} = \boldsymbol{a}_{INS,t}^{local}\Delta t$$

$$\boldsymbol{\delta\Psi}_{INS,t}^{local} = \boldsymbol{\Psi}_{KF,t-1}^{local} - \boldsymbol{\Psi}_{INS,t-1}^{local}$$

# Kalman Filter——GNSS/INS (Closed Loop)

System States:

$$\mathbf{X}_t = (\mathbf{X}_t^{local}, \mathbf{V}_t^{local}, \mathbf{\Psi}_t^{local}, \mathbf{B}_{a,t}^{body}, \mathbf{B}_{\omega,t}^{body})$$

$$\mathbf{X}_t^{local} = (x_t^{local}, y_t^{local}, z_t^{local})$$

$$\mathbf{V}_t^{local} = (vx_t^{local}, vy_t^{local}, vz_t^{local})$$

$$\mathbf{\Psi}_t^{local} = (\phi_{roll}, \theta_{pitch}, \psi_{yaw})$$

$$\mathbf{B}_{a,t}^{body} = (b_{ax,t}^{body}, b_{ay,t}^{body}, b_{az,t}^{body})$$

$$\mathbf{B}_{\omega,t}^{body} = (b_{\omega x,t}^{body}, b_{\omega y,t}^{body}, b_{\omega z,t}^{body})$$

Measurement model:

$$\Delta\mathbf{Z}_t = \mathbf{Z}_t - \mathbf{H}\mathbf{X}_t^{-}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3\times3} & \cdots & \mathbf{0} \\ \vdots & \mathbf{0} & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Propagation model:

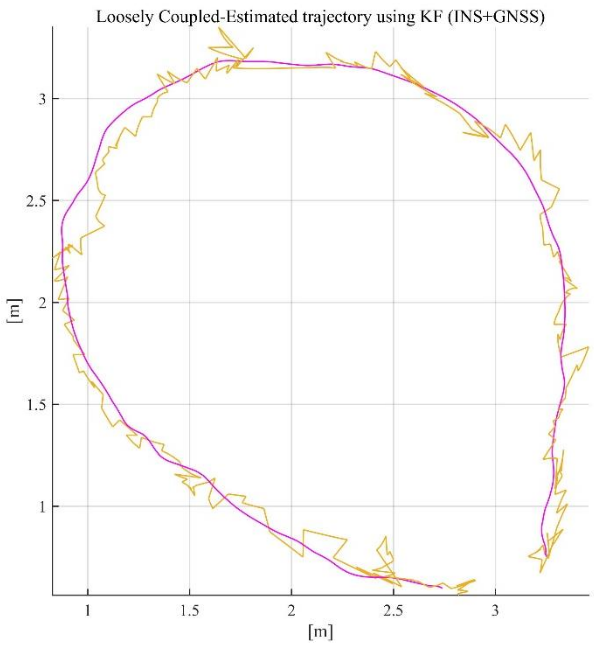
$$\mathbf{X}_t^{-} = \mathbf{F}\mathbf{X}_{t-1}^{+} + \mathbf{B}\mathbf{U}_t$$

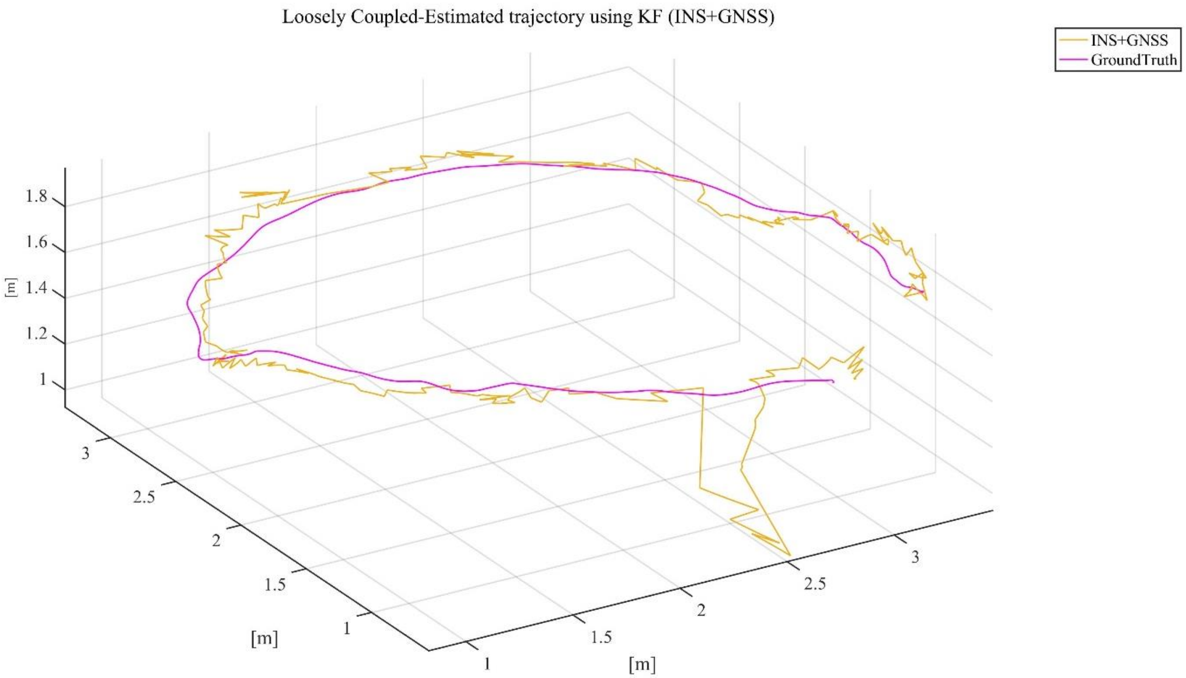
$$\mathbf{F} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{U}_t = \begin{bmatrix} \delta\mathbf{X}_{INS,t}^{local} \\ \delta\mathbf{V}_{INS,t}^{local} \\ \delta\mathbf{\Psi}_{INS,t}^{local} \\ W_{b_\omega} \\ W_{b_a} \end{bmatrix}$$

$$\mathbf{Z}_t = \mathbf{X}_{GNSS,t}^{local} = \begin{bmatrix} x_{GNSS,t}^{local} \\ y_{GNSS,t}^{local} \\ z_{GNSS,t}^{local} \end{bmatrix}$$

# Closed Loop


Loosely Coupled-Estimated trajectory using KF (INS+GNSS)
[m]
3
2.5
2
1.5
1
1
1.5
2
2.5
3
[m]


Loosely Coupled-Estimated trajectory using KF (INS+GNSS)
INS+GNSS
GroundTruth
[m]
1.8
1.6
1.4
1.2
1
3
2.5
2
1.5
1
[m]
1
1.5
2
2.5
3
[m]

# References

- Chapters 3 and 5, Paul D. Groves, *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems, 2nd Edition*, Artech House, 2013.